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On the Global Geometry of Teleparallel Gravity

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"The best that most of us can hope to achieve in physics is simply to misunderstand at a deeper level."

Wolfgang Pauli

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Resumo

A descrição teórica da gravitação teleparalela em termos de fibrados parece desencadear algumas controvérsias entre matemáticos e físicos. Acredito que o mal-entendido se deve em boa parte à falta de uma ponte entre os conceitos matemáticos rigorosos e a aplicação real desses conceitos na física. Esta tese de mestrado busca preencher essa lacuna e iluminar a estrutura geométrica da gravitação teleparalela.

Nenhuma familiaridade prévia com fibrados é assumida. Todas as ferramentas necessárias da área dos fibrados são desenvolvidas desde o início, construindo os fundamentos necessários para estudar as teorias clássicas de calibre, por um lado, e as teorias da relatividade geral modificada, por outro, com foco especial nos métodos necessários para o estudo da gravitação teleparalela.

Usando os métodos desenvolvidos, chegamos a três conclusões principais. Em primeiro lugar, um fibrado principal do grupo de translação é trivial. Em segundo lugar, uma derivada covariante compatível com a métrica e livre de curvatura num espaço-tempo simplesmente-conexo induz uma tetrada ortonormal global. Em terceiro lugar, como corolário do teorema de Geroch, um espaço-tempo simplesmente-conexo admite uma estrutura de spin se e somente se admite uma derivada covariante compatível com a métrica e livre de curvatura.

A relevância do primeiro resultado se deve aos esforços feitos pela gravitação teleparalela para descrever a gravitação usando um fibrado principal do grupo de translação. Nesta perspectiva, o primeiro resultado parece nos desencorajar da missão de descrever a gravitação usando um fibrado principal do grupo de translação, pois restringe a classe de espaços-tempos a qual o formalismo da gravitação teleparalela pode ser aplicado. O segundo resultado, entretanto, relativiza essa obstrução motivada inteiramente pela perspectiva da gravitação teleparalela. Finalmente, o terceiro resultado revela que a gravitação teleparalela não está proibindo grandes classes de espaços-tempos. Sempre que os campos de espinores são definíveis em um espaço-tempo simplesmente-conexo, também existe uma descrição em termos da gravitação teleparalela.

As novidades deste trabalho incluem uma definição e estudo completo de fibrados afins (como uma generalização de fibrados vetoriais), uma discussão independente e acessível sobre a equivalência de derivadas covariantes e sistemas de transporte paralelo em fibrados vetoriais, bem como de sua curvatura e holonomia.

Abstract

The bundle theoretic description of teleparallel gravity seems to spark some controversities between mathematicians and physicists. I believe that the misunderstanding is to a good part due to the lack of a bridge between the rigorous mathematical concepts and the actual application of these concepts in physics. This master's thesis seeks to bridge this gap and shine light on the geometric structure of teleparallel gravity.

No prior familiarity with fibre bundles is assumed. All the necessary tools from the area of fibre bundles are developed from the ground up, laying the fundament necessary in order to study classical gauge theories on the one hand, and theories of modified general relativity on the other hand, with special focus laid on the methods needed for teleparallel gravity.

Using the developed methods, we arrive at three major conclusions. First, a principal bundle of the translation group is trivial. Second, a curvature-free metric-compatible covariant derivative for a simply-connected spacetime induces a global orthonormal frame. Third, as a corollary to Geroch's theorem, a simply-connected spacetime admits a spin structure if and only if it admits a curvature-free metric-compatible covariant derivative.

The relevance of the first result is due to the efforts made by teleparallel gravity to describe gravity using a principal bundle of the translation group. In this light, the first result appears to discourage us from the mission to describe gravity using a principal bundle of the translation group since it restricts the class of spacetimes that the formalism of teleparallel gravity can be applied to. The second result, however, relativates this obstruction motivated entirely from the persepective of teleparallel gravity. Finally, the third result assures us that teleparallel gravity is not forbidding large classes of spacetimes. Whenever spinor fields are definable on a simply-connected spacetime, there also exists a description in terms of teleparallel gravity for it.

Novelties of this work include a thorough definition and study of affine bundles (as a generalization of vector bundles), a self-contained and accessible discussion of the equivalence of covariant derivatives and parallel transport systems on vector bundles, as well as of their curvature and holonomy.

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1 Revision

Modern theoretical physics immensely employs abstract mathematical concepts. This trend is not a new development. We may regard the moment that Einstein formulated his general theory of relativity in the language of state-of-the-art differential geometry as the start of a new chapter of theoretical physics. Subsequently, quantum mechanics too made great advances using methods from the field of functional analysis. In both cases, the introduction of a new level of abstraction separated the theoretical description from the experiment while allowing to describe a wide variety of different physical systems using the same fundamental laws. All in all, it is a story of great success and it is far from finished. Though, whilst the abstraction allows to describe a great variety of different physical phenomena using merely a few basic rules, it is usually not the starting point for innovation in the field of physics. Innovation in the field of physics is brought up using a more intuitive language, tied more closely to the real world.

In comparison to the two fundamental theories of general relativity and quantum mechanics, the classical field theories of the other fundamental forces were formulated in terms of abstract mathematical concepts more recently. Interestingly, the classical field theories of the fundamental forces mediated by forces share a good extent of their mathematical structure with the theory of general relativity. The main difference is that while classical field theories admit internal gauge degrees of freedom, general relativity does not admit internal gauge degrees of freedom. This is due to the very construction of general relativity as a geometric theory of gravity, bearing at its heart the experimentally uncontested assumption that the inertial and gravitational masses coincide for all particles. This assumption is called *universality*.

The main goal of teleparallel gravity is to undo this geometrization and allow a description of gravity in the absence of universality which resembles the description of the other classical field theories. In order to make progress, we have to study the field of differential geometry with special focus laid on fibre bundles, the basis of the mathematically descriptions of general relativity and classical field theories.

I will introduce all the necessary terminology of the field of fibre bundles. This is due to the fact that there is no satisfactory standard reference known to me on the subject that covers all the relevant parts. I assume the reader to be familiar with the subject of smooth manifolds. Prior familiarity with the description of general relativity in terms of pseudo-Riemannian geometry will help to motivate some of the definitions to be introduced.

Refer to the appendix A for a collection of the most important definitions from point-set topology that we will use throughout the work. In doubt, refer to an authoritative reference book such as [Mun14].

Throughout the text we will work with the following definition of a topological manifold:

Definition 1.1 (Topological manifold)**.** A *topological manifold* is a topological space $(M, \mathcal{O}_M)$ such that

1. $(M, \mathcal{O}_M)$ is locally Euclidean of dimension d for some $d \in \mathbb{N}$,
2. $(M, \mathcal{O}_M)$ is Hausdorff, and
3. one of the following equivalent conditions is met:
   - $(M, \mathcal{O}_M)$ is second countable
   - $(M, \mathcal{O}_M)$ is Lindelöf
   - $(M, \mathcal{O}_M)$ is paracompact and has at most countably many connected components[1]

[1] *Introduction to Smooth Manifolds* (2nd edition) by J. M. Lee, Page 30, Problem 1-5, Chapter 1, Section 5

As such a topological manifold is also locally compact, Lindelöf, locally path connected, normal and metrisable. Moreover, if it is connected it is also path-connected. In the case of a smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ (a topological manifold $(M, \mathcal{O}_M)$ together with a choice of smooth atlas $\mathcal{A}_M$) Whitney's approximation theorems assure that M is path-connected by smooth paths.

As discussed in chapter 10, what we intuitively understand as a *spacetime* is modelled well by a topological manifold with some additional structure, see definition 10.1. Before discussing the physics, however, we need to build a solid mathematical understanding of the necessary concepts. This is the purpose of the chapters 2 to 9.

2 Fibre bundles

Loosely speaking a *fibre bundle* is a smooth manifold that can locally be expressed as a product of manifolds, but in general not globally. The following definition turns this notion rigorous:

Definition 2.1 (Fibre bundle (over a smooth manifold M with typical fibre F))**.** Let $(E, \mathcal{O}_E, \mathcal{A}_E)$, $(M, \mathcal{O}_M, \mathcal{A}_M)$, $(F, \mathcal{O}_F, \mathcal{A}_F)$ be three smooth manifolds and $\pi\colon E \to M$ a smooth surjective submersion.

The structure $\pi\colon E \to M$ is said to be a **fibre bundle over** M **with typical fibre** F if for any point $p \in M$ there exists a neighbourhood U of p and a smooth map, called **bundle chart**,

$$\alpha\colon \pi^{-1}[U] \to F$$

such that

$$(\pi, \alpha)\colon \pi^{-1}(U) \to U \times F$$

is a diffeomorphism. We say that $E_p \coloneqq \pi^{-1}[\{p\}]$ is the **fibre over** $p \in M$.

Definition 2.2 (Bundle atlas)**.** Let $\pi\colon E \to M$ be a fibre bundle over M with typical fibre F. A **bundle atlas** on E is a collection of bundle charts, i.e., a subset $\mathcal{B} \subseteq \bigcup \{C^\infty(A, F) \mid A \subseteq E\}$ satisfying:

1. $\{\pi[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ is an open cover of the base space M.
2. $\forall \alpha \in \mathcal{B}\colon\ \mathrm{Dom}(\alpha) = \pi^{-1}[\pi[\mathrm{Dom}(\alpha)]]$
3. $\forall \alpha \in \mathcal{B}\colon (\pi, \alpha)\colon\ \mathrm{Dom}(\alpha) \to \pi[\mathrm{Dom}(\alpha)] \times F$ is a diffeomorphism

Remark 2.1 (Existence of a bundle atlas). The second part of the definition of a fibre bundle precisely guarantees the existence of a bundle atlas for it. In fact, it is sufficient to require the projection $\pi\colon E \to M$ to be a smooth surjection. The existence of a bundle atlas ensures that it is a submersion.

Definition 2.3 (Transition functions of a bundle atlas)**.** Let $\mathcal{B}$ be a bundle atlas of a fibre bundle $\pi\colon E \to M$ with typical fibre F. For every two bundle charts $\alpha, \beta \in \mathcal{B}$ with non-empty domain intersection $\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta) \neq \emptyset$, we can define the **transition function** from α to β

$$\tilde{\rho}_{\beta\alpha}\colon \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)\right] \to \mathrm{Diff}(F), \quad p \mapsto \beta|_{E_p} \circ \alpha|_{E_p}^{-1}. \tag{2.1}$$

The collection $\{\tilde{\rho}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{B}\}$ is called the collection of **transition functions of the bundle atlas** $\mathcal{B}$.

Remark 2.2 (Transition functions of a bundle atlas satisfy cocycle conditions). The transition functions $\{\tilde{\rho}_{\beta\alpha} \mid \alpha, \beta \in \mathcal{B}\}$ of a bundle atlas $\mathcal{B}$ satisfy the following cocycle conditions:

$$\forall \alpha \in \mathcal{B}\colon \forall p \in \pi[\mathrm{Dom}(\alpha)]\colon \qquad \tilde{\rho}_{\alpha\alpha}(p) = \mathrm{id}_F, \tag{2.2}$$

$$\forall \alpha, \beta, \gamma \in \mathcal{B}\colon \forall p \in \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \cap \pi[\mathrm{Dom}(\gamma)]\colon \qquad \tilde{\rho}_{\gamma\alpha}(p) = \tilde{\rho}_{\gamma\beta}(p) \circ \tilde{\rho}_{\beta\alpha}(p). \tag{2.3}$$

Theorem 2.1 (Equipping a set with the structure of a fibre bundle using a bundle atlas)**.** *Suppose we are given a set E, two smooth manifolds $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(F, \mathcal{O}_F, \mathcal{A}_F)$, and a surjective map $\pi\colon E \to M$.*

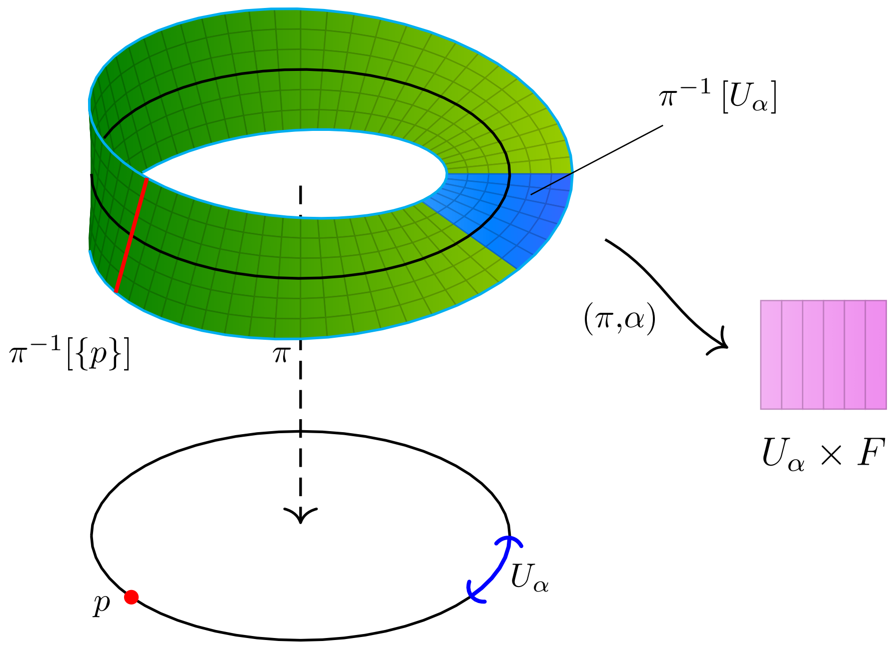

Figure 2.1: The Möbius strip over a circle is an example of a fibre bundle.

Suppose further that we are given a subset $\mathcal{B} \subseteq \bigcup \{A \to F \mid A \subseteq E\}$ *satisfying:*

1. $\{\pi[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ *is an open cover of* M
2. $\forall \alpha \in \mathcal{B}\colon\ \mathrm{Dom}(\alpha) = \pi^{-1}[\pi[\mathrm{Dom}(\alpha)]]$
3. $\forall \alpha \in \mathcal{B}\colon (\pi, \alpha)\colon\ \mathrm{Dom}(\alpha) \to \pi[\mathrm{Dom}(\alpha)] \times F$ *is a bijection*
4. $\forall \alpha, \beta \in \mathcal{B}\colon (\pi, \beta) \circ (\pi, \alpha)^{-1}\colon \pi[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)] \times F \to \pi[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)] \times F$ *is a diffeomorphism*

Then we can equip E *with a unique topology* $\mathcal{O}_E$ *and a unique smooth structure* $[\mathcal{A}_E]_\infty$ *such that* $\pi\colon E \to M$ *is a fibre bundle with bundle atlas* $\mathcal{B}$*.*

Proof 2.1. We first equip E with a topology $\mathcal{O}_E$ that makes $(E, \mathcal{O}_E)$ a topological manifold.

Subproof (Which topology $\mathcal{O}_E$ to choose for E?). We might start with the initial topology $\tau(\pi) = \left\{\pi^{-1}[W] \mid W \in \mathcal{O}_M\right\}$ on E with respect to the surjection $\pi\colon E \to M$, thus making $\pi\colon E \to M$ a continuous map. Note that for any $\alpha \in \mathcal{B}$, we have that $\mathrm{Dom}(\alpha) \in \tau(\pi)$. Since $\{\pi[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ is an open cover of M, we know that $\{\mathrm{Dom}(\alpha) \mid \alpha \in \mathcal{B}\}$ is an open cover of E with respect to $\tau(\pi)$.

However, this topology is not fine enough for it fails to render the local trivialization (π, α) into a homeomorphism for each $\alpha \in \mathcal{B}$. We thus refine our topology $\mathcal{O}_E$ on E starting from the topological basis

$$\mathcal{S} = \left\{(\pi, \alpha)^{-1}[V] \mid \alpha \in \mathcal{B} \wedge V \in \mathcal{O}_{\pi[\mathrm{Dom}(\alpha)] \times F}\right\} \subseteq \mathcal{P}(E). \tag{2.4}$$

By design, this topology renders $(\pi, \alpha)\colon\ \mathrm{Dom}(\alpha) \to \pi[\mathrm{Dom}(\alpha)] \times F$ into a homeomorphism for each $\alpha \in \mathcal{B}$. In the following we will prove that $(E, \mathcal{O}_E)$ is a topological manifold.

Subproof ($(E, \mathcal{O}_E)$ is locally Euclidean). Let $a \in E$. Since $\{\pi\,[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ is an open cover of M, there exists $\alpha \in \mathcal{B}$ such that $a \in \mathrm{Dom}(\alpha)$. Now choose a chart $x \in \mathcal{A}_M$ at $\pi(a)$ and a chart $\xi \in \mathcal{A}_F$ at $\alpha(a)$. Recall that their Cartesian product $x \times \xi\colon\ \mathrm{Dom}(x) \times \mathrm{Dom}(\xi) \to \mathbb{R}^{\dim(M)+\dim(F)}$ is a diffeomorphism onto its image. Since $\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x) \in \mathcal{O}_M$ and $\mathrm{Dom}(\xi) \in \mathcal{O}_F$, the preimage

$$(\pi, \alpha)^{-1}\left[(\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)) \times \mathrm{Dom}(\xi)\right] \tag{2.5}$$

is open in $(E, \mathcal{O}_E)$. Furthermore, the map

$$\begin{aligned}(x \times \xi) \circ (\pi, \alpha) \colon (\pi, \alpha)^{-1} \left[(\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)) \times \mathrm{Dom}(\xi) \right] \\ \to x[\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)] \times \xi[\mathrm{Dom}(\xi)]\end{aligned} \tag{2.6}$$

is a homeomorphism onto its image as the composition of two homeomorphisms onto their images.

Subproof ($(E, \mathcal{O}_E)$ is Hausdorff). Let $a, b \in E$ such that $a \neq b$. If a and b happen to lie in distinct fibres, then $\pi(a) \neq \pi(b)$. Since M is Hausdorff, there exist open subsets $W_a, W_b \in \mathcal{O}_M$ such that $a \in W_a, b \in W_b$ and $W_a \cap W_b = \emptyset$. Since $\pi \colon (E, \mathcal{O}_E) \to (M, \mathcal{O}_M)$ is continuous, we have that $\pi^{-1}[W_a], \pi^{-1}[W_b] \in \mathcal{O}_E$. We found open subsets $\pi^{-1}[W_a] \in \mathcal{O}_E$ and $\pi^{-1}[W_b] \in \mathcal{O}_E$ that satisfy $a \in \pi^{-1}[W_a]$, $b \in \pi^{-1}[W_b]$ and $\pi^{-1}[W_a] \cap \pi^{-1}[W_b] = \emptyset$.

If a and b happen to lie in the same fibre, i.e. $\pi(a) = \pi(b)$, there exists a bundle chart α that contains both a and b in its domain. This is because $\{\pi[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ is an open cover of M and $\mathrm{Dom}(\alpha) = \pi^{-1}[\pi[\mathrm{Dom}(\alpha)]]$ for each $\alpha \in \mathcal{B}$. Since $(\pi, \alpha) \colon \mathrm{Dom}(\alpha) \to \pi[\mathrm{Dom}(\alpha)] \times F$ is a homeomorphism and since $\pi[\mathrm{Dom}(\alpha)] \times F$ is Hausdorff, there exist $V_a, V_b \in \mathcal{O}_{\pi[\mathrm{Dom}(\alpha)] \times F}$ such that $(\pi(a), \alpha(a)) \in V_a$, $(\pi(b), \alpha(b)) \in V_b$ and $V_a \cap V_b = \emptyset$. Finally, we have $a \in (\pi, \alpha)^{-1}[V_a] \in O_E$ and $b \in (\pi, \alpha)^{-1}[V_b] \in O_E$ such that $(\pi, \alpha)^{-1}[V_a] \cap (\pi, \alpha)^{-1}[V_b] = \emptyset$. We conclude that $(E, \mathcal{O}_E)$ is Hausdorff.

Subproof ($(E, \mathcal{O}_E)$ is second countable). Suppose that $\mathcal{A}_F$ was chosen to be a countable smooth atlas, which always exists since $(F, \mathcal{O}_F)$ is Lindelöf, cf. definition 1.1. Note that

$$\mathcal{U} := \{\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x) \mid (\alpha, x) \in \mathcal{B} \times \mathcal{A}_M\} \tag{2.7}$$

is an open cover of $(M, \mathcal{O}_M)$. Since $(M, \mathcal{O}_M)$ is Lindelöf, there exists a countable subcover $\mathcal{U}' \subseteq \mathcal{U}$. Due to the axiom of choice, there exists a choice function $c \colon \mathcal{U}' \to \mathcal{B} \times \mathcal{A}_M$ with the property that for every $U \in \mathcal{U}'$ the value $(\alpha, x) = c(U)$ at U satisfies $U = \pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)$. Note that the image $c[\mathcal{U}'] \subseteq \mathcal{B} \times \mathcal{A}_M$ is countable. We now claim that

$$\begin{aligned}\mathcal{A}_E = \Big\{ (x \times \xi) \circ (\pi, \alpha) \colon (\pi, \alpha)^{-1} \left[(\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)) \times \mathrm{Dom}(\xi) \right] \\ \to \mathbb{R}^{\mathrm{Dim}(M) + \mathrm{Dim}(F)} \mid (\alpha, x) \in c[\mathcal{U}'] \wedge \xi \in \mathcal{A}_F \Big\}\end{aligned} \tag{2.8}$$

is a countable atlas of $(E, \mathcal{O}_E)$. It is countable as the image of the Cartesian product of the countable sets $c[\mathcal{U}']$ and $\mathcal{A}_F$, relying on the fact that the Cartesian product of countable sets is countable. It is left to show that

$$\left\{ (\pi, \alpha)^{-1} \left[(\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)) \times \mathrm{Dom}(\xi) \right] \mid (\alpha, x) \in c[\mathcal{U}'] \wedge \xi \in \mathcal{A}_F \right\} \tag{2.9}$$

covers $(E, \mathcal{O}_E)$. Let $a \in E$. Since $\{\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x) \mid (\alpha, x) \in c[\mathcal{U}']\}$ covers $(M, \mathcal{O}_M)$ and $\{\mathrm{Dom}(\xi) \mid \xi \in \mathcal{A}_F\}$ covers $(F, \mathcal{O}_F)$, there exist $(\alpha, x) \in c[\mathcal{U}']$ and $\xi \in A_F$ such that $\pi(a) \in \pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)$ and $\alpha(a) \in \mathrm{Dom}(\xi)$. Hence $a \in (\pi, \alpha)^{-1} \left[(\pi[\mathrm{Dom}(\alpha)] \cap \mathrm{Dom}(x)) \times \mathrm{Dom}(\xi) \right]$.

Subproof (Smooth structure). It remains to check that the provided atlas $\mathcal{A}_E$ is a smooth atlas for $(E, \mathcal{O}_E)$. It suffices to show that the chart transition maps are diffeomorphisms in the Euclidean sense. To this end, let $(x \times \xi) \circ (\pi, \alpha), (y \times \zeta) \circ (\pi, \beta) \in \mathcal{A}_E$ be charts and consider the transition map:

$$\begin{aligned}(y \times \zeta) \circ (\pi, \beta) \circ ((x \times \xi) \circ (\pi, \alpha))^{-1} : x[\pi[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)] \cap \mathrm{Dom}(x) \cap \mathrm{Dom}(y)] \times \xi[\mathrm{Dom}(\xi) \cap \mathrm{Dom}(\zeta)] \\ \to y[\pi[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)] \cap \mathrm{Dom}(x) \cap \mathrm{Dom}(y)] \times \zeta[\mathrm{Dom}(\xi) \cap \mathrm{Dom}(\zeta)]\end{aligned}$$

On the entire domain, we have that:

$$(y \times \zeta) \circ (\pi, \beta) \circ ((x \times \xi) \circ (\pi, \alpha))^{-1} = (y \times \zeta) \circ \left((\pi, \beta) \circ (\pi, \alpha)^{-1} \right) \circ (x \times \xi)^{-1} \tag{2.10}$$

However, by requirement, $(\pi, \beta) \circ (\pi, \alpha)^{-1}$ is an diffeomorphism and therefore, $(x \times \xi)$ and $(y \times \zeta)$ being charts of $\pi[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)] \times F$, the above map is a diffeomorphism in the Euclidean sense. Note that for every $\alpha \in \mathcal{B}$, $x \in \mathcal{A}_M$ and $\xi \in \mathcal{A}_F$, the map $(x \times \xi) \circ (\pi, \alpha)$ is a chart of E that is smoothly compatible with $\mathcal{A}_E$.

Subproof (Fibre Bundle). We already know that $\pi\colon E \to M$ is a continuous surjection. First, observe that $\pi\colon E \to M$ is smooth at every point $a \in E$. For, if $(x \times \xi) \circ (\pi, \alpha) \in \mathcal{A}_E$ is a chart of E at a, we have the equality of maps on the domain in question

$$x \circ \pi \circ (\pi, \alpha)^{-1} \circ (x \times \xi)^{-1} = \mathrm{pr}_1, \tag{2.11}$$

which is smooth in the Euclidean sense. Likewise, any bundle chart $\alpha \in \mathcal{B}$ is smooth at every point $a \in \mathrm{Dom}(\alpha)$. For, if $x \in \mathcal{A}_M$ and $\xi \in \mathcal{A}_F$ are charts of M and F, respectively, then $(x \times \xi) \circ (\pi, \alpha)$ is a chart of E as we had concluded in the preceding item. On the domain in question, we have the equality of maps

$$\xi \circ \alpha \circ (\pi, \alpha)^{-1} \circ (x \times \xi)^{-1} = \mathrm{pr}_2, \tag{2.12}$$

which, once more, is smooth in the Euclidean sense. This also proves that $(\pi, \alpha)\colon \mathrm{Dom}(\alpha) \to \pi[\mathrm{Dom}(\alpha)] \times F$ is smooth. Its inverse $(\pi, \alpha)^{-1}\colon \pi[\mathrm{Dom}(\alpha)] \times F \to \mathrm{Dom}(\alpha)$ is smooth due to the fact that

$$(x \times \xi) \circ (\pi, \alpha) \circ (\pi, \alpha)^{-1} \circ (x \times \xi)^{-1} = \mathrm{id}, \tag{2.13}$$

is smooth in the Euclidean sense, too. This concludes the proof.

Example 2.1 (The tangent bundle of a smooth manifold)**.** The above theorem can be used to prove that for every smooth manifold M we can canonically construct a fibre bundle

$$\pi\colon TM \to M, v \mapsto p \text{ such that } v \in T_pM \tag{2.14}$$

whose total space $TM := \{T_pM \mid p \in M\}$ is the (disjoint) union of the tangent spaces T_pM at all points $p \in M$. Suppose we are given a smooth atlas $\mathcal{A}_M$ for M. It is straightforward to prove that the collection

$$\mathcal{B} := \left\{x_*\colon \pi^{-1}[\mathrm{Dom}(x)] \to \mathbb{R}^{\mathrm{Dim}(M)} \mid x \in \mathcal{A}_M\right\} \tag{2.15}$$

satisfies the hypothesis of theorem 2.1. We call $\pi\colon TM \to M$ the **tangent bundle of** M.

Example 2.2 (The differential bundle of a fibre bundle)**.** Suppose $\pi\colon E \to M$ is a fibre bundle with typical fibre F and bundle atlas $\mathcal{B}$. The differential $\pi_*\colon TE \to TM$ of π is a fibre bundle with typical fibre TF and bundle atlas

$$\mathcal{B}_{\pi_*} := \{\alpha_*\colon T\,\mathrm{Dom}(\alpha) \to TF \mid \alpha \in \mathcal{B}\}\,, \tag{2.16}$$

by theorem 2.1.

Example 2.3 (The pullback of a fibre bundle along a smooth map)**.** Let $\pi\colon E \to M$ be a fibre bundle with typical fibre F and bundle atlas $\mathcal{B}$. Let $f\colon N \to M$ be a smooth map. Define the set

$$f^*E := \{(p, a) \in N \times E \mid f(p) = \pi(a)\}\,. \tag{2.17}$$

The map $\Pi\colon f^*E \to N, (p, a) \mapsto p$ is a fibre bundle with bundle atlas

$$f^*\mathcal{B} := \left\{\alpha \circ \mathrm{pr}_2\colon \Pi^{-1}\left[f^{-1}[\pi[\mathrm{Dom}(\alpha)]]\right] \to F \mid \alpha \in \mathcal{B}\right\}. \tag{2.18}$$

Definition 2.4 (Section of a fibre bundle)**.** Let $\pi\colon E \to M$ be a fibre bundle over M with typical fibre F. A **section** is a smooth map $s\colon M \to E$ such that $\pi \circ s\colon M \to M$ coincides with the identity map $\mathrm{id}_M\colon M \to M$, i.e., such that the following diagram commutes:

$$\begin{tikzcd} & E \arrow[dr, "\pi"] & \\ M \arrow[ur, "s"] \arrow[rr, "\mathrm{id}_M"'] & & M \end{tikzcd}$$

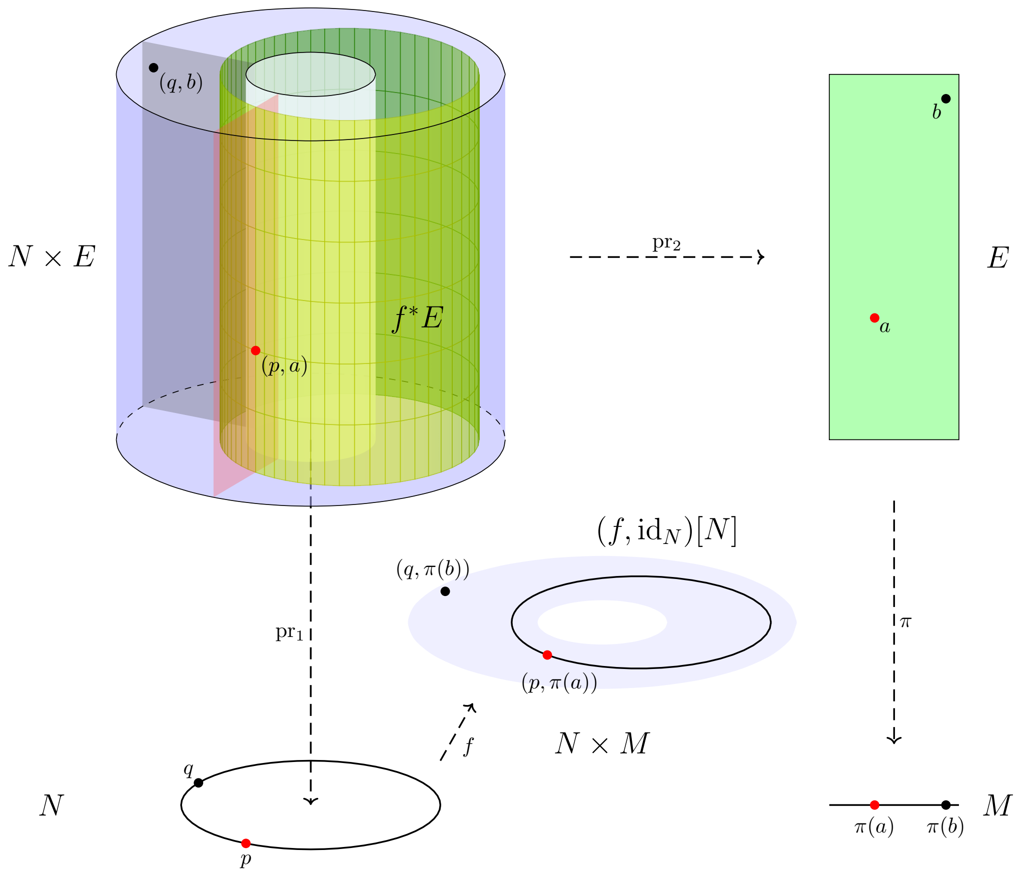

Figure 2.2: An exemplary construction of a pullback bundle along a map $f\colon S^1 \to M$.

We denote the set of sections of $\pi\colon E \to M$ by $\Gamma(\pi)$, or - if no confusion is possible - by $\Gamma(E)$.

Remark 2.3. We will also refer to a section $s\colon M \to E$ of a fibre bundle as **global section**. The motivation behind this terminology relies on the following context. Let $U \subseteq M$ be an open subset of M. Since $\pi\colon E \to M$ is continuous, the set $E|_U := \pi^{-1}[U]$ is an open set of E. Then $\pi|_{E|_U}\colon E|_U \to U$ is a fibre bundle as well. A section of $\pi|_{E|_U}\colon E|_U \to U$ is called a **local section** of the fibre bundle $\pi\colon E \to M$. Note that the restriction of a global section to any open set of M is a local section. As we will see, a fibre bundle might not admit a global section. The existence of local sections, on the contrary, is guaranteed by the existence of a bundle atlas.

Definition 2.5. Let $\pi\colon E \to M$ be a fibre bundle and $f\colon M \to N$ a smooth map. A smooth map $s\colon N \to E$ that satisfies $\pi \circ s = f$ is said to be a **section along** f, denoted by $s \in \Gamma_\gamma(E)$.

Remark 2.4. Let $\pi\colon E \to M$ be a fibre bundle and $f\colon M \to N$ a smooth map. There is a one-to-one correspondence between sections along f and sections of the pullback bundle f^*E.

Example 2.4. Given a curve $\gamma\colon \mathbb{R} \to M$ on a smooth manifold, the map

$$\mathbb{R} \to TM, \quad t \mapsto \dot{\gamma}(t) \tag{2.19}$$

that attributes to every $t \in \mathbb{R}$ the tangent vector $\dot{\gamma}(t)$ of the curve γ at t can be understood as a section of TM along γ or as a section of the pullback bundle $\gamma^*(TM)$.

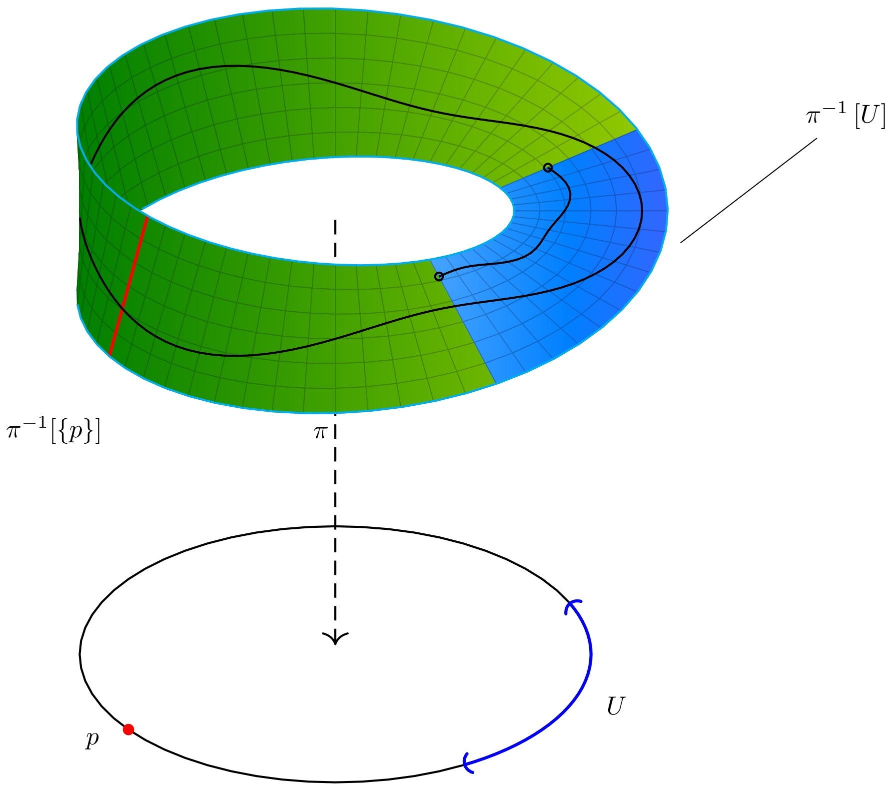

Figure 2.3: A global and a local section of a fibre bundle.

Definition 2.6 (Bundle morphism)**.** Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be two fibre bundles. Suppose we are given two smooth maps $\Phi\colon E \to E'$ and $\varphi\colon M \to M'$.

We say that Φ is a **bundle morphism along** φ, or simply that (Φ, φ) is a **bundle morphism from** $\pi\colon E \to M$ **to** $\pi'\colon E' \to M'$, if the following diagram commutes:

$$\begin{tikzcd} E \arrow[r, "\Phi"] \arrow[d, "\pi"'] & E' \arrow[d, "\pi'"] \\ M \arrow[r, "\varphi"'] & M' \end{tikzcd}$$

Put differently, a **bundle morphism** $\Phi\colon E \to E'$ **along** $\varphi\colon M \to M'$ maps each fibre $E_p = \pi^{-1}[\{p\}]$ onto the fibre $E'_{\varphi(p)} = \pi'^{-1}[\{\varphi(p)\}]$ in a smooth fashion.

Definition 2.7 (Bundle isomorphism)**.** A bundle morphism (Φ, φ) from $\pi\colon E \to M$ to $\pi'\colon E' \to M'$ is said to be a **bundle isomorphism** if $\Phi\colon E \to E'$ and $\varphi\colon M \to M'$ are diffeomorphisms, i.e., if $(\Phi^{-1}, \varphi^{-1})$ is defined and a bundle morphism from $\pi'\colon E' \to M'$ to $\pi\colon E \to M$.

Definition 2.8 (Trivial fibre bundle)**.** We say that a fibre bundle $\pi\colon E \to M$ with typical fibre F is **trivial** if it admits a *global* bundle chart, or equivalently, if there exists a bundle isomorphism (Φ, φ) from $\pi\colon E \to M$ to $\mathrm{pr}_1\colon M \times F \to F, (p, f) \mapsto p$.

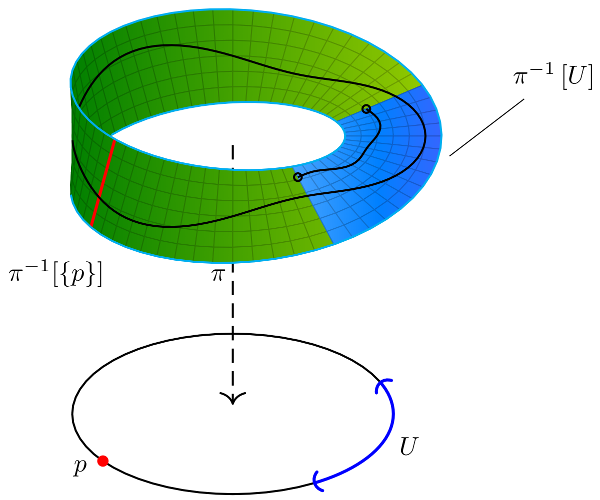

Figure 2.4: The Möbius strip is an example of a non-trivial fibre bundle. It cannot be expressed as a Cartesian product of two smooth manifolds.

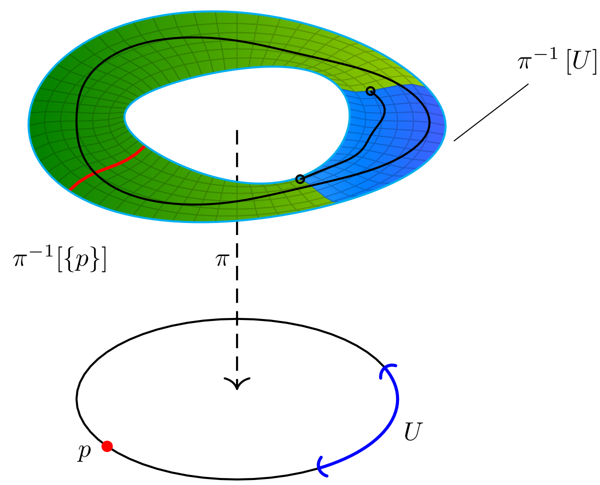

Figure 2.5: The cylinder is an example of a trivial fibre bundle. It can be expressed as the Cartesian product $S^1 \times \mathbb{R}$.

3 Fibre bundles with effective structure group action

Fibre bundles for themselves are rather simple structures. They become more interesting for applications in physics once we equip them with more additional structure. In this chapter we set the starting point of this journey and get in contact with some more specific and more useful examples of fibre bundles.

Throughout most of the first part of this chapter denote by $(G, \mathcal{O}_G, \mathcal{A}_G, \bullet)$ be a Lie group and by $(M, \mathcal{O}_M, \mathcal{A}_M)$ a smooth manifold. Before continuing make sure that you are familiar with concepts of appendix C.

Definition 3.1 (Lie group left action (on a smooth manifold))**.** A smooth left action $\triangleright\colon G \times M \to M$ is called a **left G-action on M**.

Definition 3.2 (Lie group right action (on a smooth manifold))**.** A smooth right action $\triangleleft\colon M \times G \to M$ is called a **right G-action on M**.

Definition 3.3 (Lie group action compatible bundle atlas)**.** Let $\triangleright\colon G \times F \to F$ be an effective Lie group left action. A bundle atlas $\mathcal{B}$ of a fibre bundle $\pi\colon E \to M$ over M with typical fibre F is said to be **$(G, \triangleright)$-compatible** if for any two bundle charts $\alpha, \beta \in \mathcal{B}$ with non-empty overlap $\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta) \neq \emptyset$ there exists a smooth map $\rho_{\beta\alpha}\colon \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)\right] \to G$ such that for any $p \in \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)\right]$ and $\xi \in F$ it holds that

$$\left(\beta|_{E_p} \circ \alpha|_{E_p}{}^{-1}\right)(\xi) = \beta \circ (\pi, \alpha)^{-1}(p, \xi) = \rho_{\beta\alpha}(p) \triangleright \xi. \tag{3.1}$$

In terms of the transition function $\tilde{\rho}_{\beta\alpha}\colon \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)\right] \to \mathrm{Diff}(F)$ the above conditions says that for any $p \in \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\beta)\right]$ it holds that $\tilde{\rho}_{\beta\alpha}(p) = \rho_{\beta\alpha}(p)\triangleright$. In this light, it is natural to call $\rho_{\beta\alpha}$ a **(G-valued) transition function** of $\mathcal{B}$.

We say that two $(G, \triangleright)$-compatible bundle atlases $\mathcal{B}$ and $\mathcal{B}'$ are **$(G, \triangleright)$-equivalent** if their union $\mathcal{B} \cup \mathcal{B}'$ is also a $(G, \triangleright)$-compatible bundle atlas. An equivalence class of $(G, \triangleright)$-equivalent $(G, \triangleright)$-compatible bundle atlases $[\mathcal{B}]_{(G,\triangleright)}$ is said to be a **$(G, \triangleright)$-bundle structure** for the fibre bundle $\pi\colon E \to M$.

Proposition 3.1 (G-valued transition functions satisfy cocycle conditions)**.** *The G-valued transition functions $\{\rho_{\beta\alpha} \mid \beta, \alpha \in \mathcal{B}\}$ of a $(G, \triangleright)$-compatible bundle atlas satisfy the following cocycle conditions:*

$$\forall \alpha \in \mathcal{B}\colon \forall p \in \pi[\mathrm{Dom}(\alpha)]\colon \qquad \rho_{\alpha\alpha}(p) = e_G, \tag{3.2}$$

$$\forall \alpha, \beta, \gamma \in \mathcal{B}\colon \forall p \in \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \cap \pi[\mathrm{Dom}(\gamma)]\colon \qquad \rho_{\gamma\alpha}(p) = \rho_{\gamma\beta}(p) \bullet \rho_{\beta\alpha}(p). \tag{3.3}$$

Proof 3.1. Recall Remark 2.2 (Transition functions of a bundle atlas satisfy cocycle conditions). The second part (equation (3.3)) is readily verified given that $\triangleright\colon G \times F \to F$ is a left action. The first part (equation (3.2)), on the contrary, does not follow from this fact. It instead relies on the action $\triangleright\colon G \times F \to F$ being effective.

Remark 3.1 (Lie group action bundle structure). The last part of Definition 3.3 (Lie group action compatible bundle atlas) indeed establishes an equivalence relation on the set of $(G, \triangleright)$-compatible bundle atlases of a fibre bundle $\pi\colon E \to M$ with given effective Lie group left action $\triangleright\colon G \times F \to F$, thus justifying well-definedness of $(G, \triangleright)$-bundle structures.

Remark 3.2 (Not every bundle atlas is Lie group action compatible). In the context of finite-dimensional manifolds, the diffeomorphism group $\mathrm{Diff}(F)$ of an at least 1-dimensional non-empty smooth manifold F does not qualify as a Lie group, for it fails to be finite-dimensional.[2] It is thus true that every fibre bundle $\pi\colon E \to M$ over a non-empty manifold M and with typical fibre F admits a bundle atlas that fails to be $(G, \triangleright)$-compatible for any Lie group G and effective Lie group left action $\triangleright\colon G \times F \to F$. Take as an example the maximal bundle atlas.

Remark 3.3 (Analogy between smooth structures and bundle structures). Recall that one of the conditions of Definition 2.1 (Fibre bundle (over a smooth manifold M with typical fibre F)) was precisely the existence of a bundle atlas. However, we do not need to provide a distinguished bundle atlas in order to define a fibre bundle. The existence of some bundle atlas ensures the existence of a unique maximal bundle atlas that contains every possible bundle atlas as a subset. This is due to the fact that the union of any two bundle atlases is also a bundle atlas. Observe that the same rationale applies to atlases of topological manifolds. Whenever a topological space qualifies as a topological manifold, there exist atlases for it. Moreover, all of them are subsets of a unique maximal atlas.

The situation changes fundamentally as soon as we look at smooth manifolds. In order to make a topological manifold into a smooth manifold we have to provide additional structure, we do this by restricting ourselves to some maximal smooth atlas. In full analogy we can make a fibre bundle into a **fibre bundle with effective structure group** by restricting ourselves to some maximal $(G, \triangleright)$-compatible bundle atlas, thus providing an additional $(G, \triangleright)$-bundle structure.

Definition 3.4 (Fibre bundle with effective structure group action). Let $\pi\colon E \to M$ be a fibre bundle over M with typical fibre F and let $\triangleright\colon G \times F \to F$ be an effective Lie group left action.

The fibre bundle $\pi\colon E \to M$ together with a $(G, \triangleright)$-bundle structure $[\mathcal{B}]_{(G,\triangleright)}$ is said to be a **fibre bundle with effective structure group action** $\triangleright$. The group G is then said to be the **structure group** of $\pi\colon E \to M$ with its bundle atlas $\mathcal{B}$.

Whenever we talk about a **fibre bundle** $\pi\colon E \to M$ **with effective structure group action** $\triangleright$ it often comes unsaid that we provide a $(G, \triangleright)$-compatible bundle atlas $\mathcal{B}$ that fixes the $(G, \triangleright)$-bundle structure.

Definition 3.5 (Structure group reduction of bundle atlas). Let $\pi\colon E \to M$ be a fibre bundle with effective effective structure group action $\triangleright\colon G \times F \to F$ over M with typical fibre F and with maximal $(G, \triangleright)$-compatible bundle atlas $\mathcal{B}$. Let $H \subseteq G$ be a Lie subgroup of G.

A subset $\mathcal{B}' \subseteq \mathcal{B}$ that qualifies as a $(H, \triangleright|_H)$-compatible bundle atlas for $\pi\colon E \to M$, where $\triangleright|_H\colon H \times F \to F, (h, \xi) \mapsto h \triangleright \xi$ is the restriction of $\triangleright$ to H, is said to be a **structure group reduction from** G **to** H of the $(G, \triangleright)$-compatible bundle atlas $\mathcal{B}$.

We will shortly encounter three special cases. While in general there does not exist a specialisation of bundle morphisms to fibre bundles with effective structure group action, there does for bundle isomorphisms.

Definition 3.6 (Isomorphism of fibre bundles with effective structure group action). Let $\pi\colon E \to M$ and $\pi'\colon E' \to M$ be two fibre bundles with effective structure group action $\triangleright\colon G \times F \to F$ over M with typical fibre F and with $(G, \triangleright)$-compatible bundle atlases $\mathcal{B}$ and $\mathcal{B}'$, respectively.

Suppose we are given a bundle isomorphism $\Phi : E \to E'$ along $\mathrm{id}_M\colon M \to M$. We say that Φ is a **bundle isomorphism of fibre bundles over** M **with effective structure group action** $\triangleright\colon G \times F \to F$ if for any two bundle charts $\alpha \in \mathcal{B}$ and $\alpha' \in \mathcal{B}'$ there exists a smooth map $\tau_{\alpha'\alpha}\colon \pi\left[\mathrm{Dom}(\alpha)\right] \cap \pi'\left[\mathrm{Dom}(\alpha')\right] \to G$ such that

$$\forall p \in \pi\left[\mathrm{Dom}(\alpha) \cap \mathrm{Dom}(\alpha')\right] : \forall \xi \in F: \quad \alpha' \circ \Phi \circ (\pi, \alpha)^{-1}(p, \xi) = \tau_{\alpha'\alpha}(p) \triangleright \xi. \tag{3.4}$$

[2] Note the correspondence between the diffeomorphism group $\mathrm{Diff}(F)$ and the (infinite-dimensional) Lie algebra of vector fields $\Gamma(TF)$ over $\mathbb{R}$.

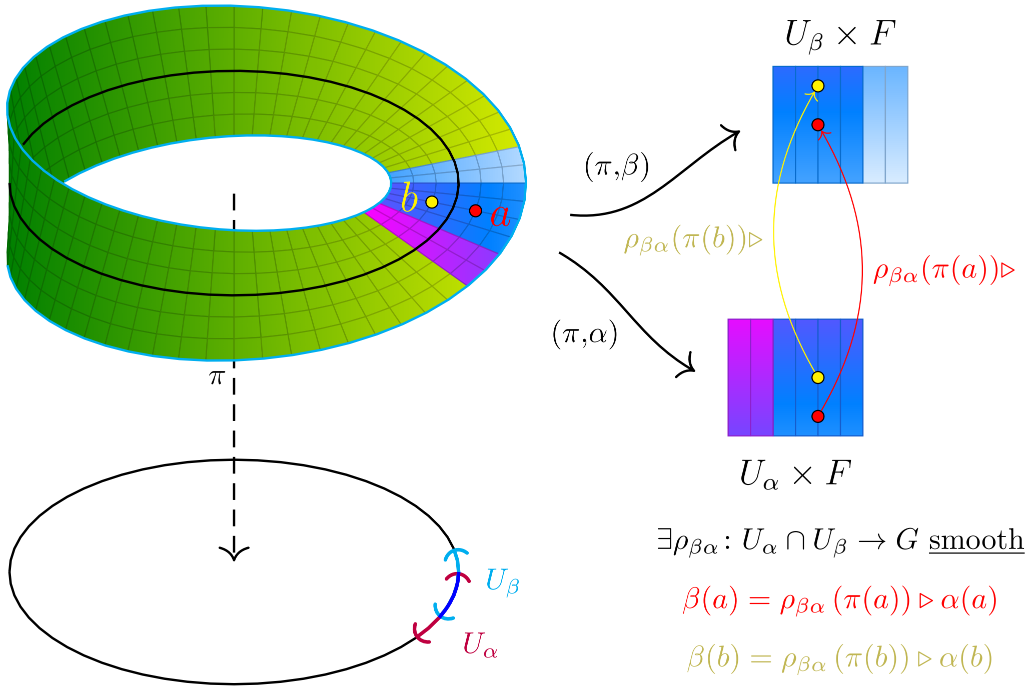

Figure 3.1: The Möbius strip may be understood as a fibre bundle with effective structure group action. As we will see, there exist bundle atlases which are compatible with respect to distinct Lie group left actions.

Proposition 3.2. *Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be two fibre bundles over M with typical fibre F and effective Lie group left action $\triangleright\colon G \times F \to F$.*

Let $\mathcal{I}$ be such that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M with the property that there exist a $(G,\triangleright)$-compatible bundle atlas $\mathcal{B}_1 = \left\{\alpha_i^1\colon {\pi_1}^{-1}[U_i] \to U_i \times F \mid i \in \mathcal{I}\right\}$ for $\pi_1\colon E_1 \to M$ and a $(G,\triangleright)$-compatible bundle atlas $\mathcal{B}_2 = \left\{\alpha_i^2\colon {\pi_2}^{-1}[U_i] \to U_i \times F \mid i \in \mathcal{I}\right\}$ for $\pi_2\colon E_2 \to M$.

Then $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ are bundle isomorphic as fibre bundles over M with effective structure group action $\triangleright\colon G \times F \to F$ if and only if there exists a family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$ that relates the transition functions $\left\{\rho^1_{ij}\colon U_i \cap U_j \to G \mid i,j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\right\}$ of $\mathcal{B}_1$ to the transition functions $\left\{\rho^2_{ij}\colon U_i \cap U_j \to G \mid i,j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\right\}$ of $\mathcal{B}_2$ through:

$$\forall i,j \in \mathcal{I}\colon \forall p \in U_i \cap U_j\colon \quad \nu_i(p) \bullet \rho^1_{ij}(p) = \rho^2_{ij}(p) \bullet \nu_j(p) \tag{3.5}$$

Proof 3.2. We start off by proving the existence of an isomorphism of fibre bundles over M with effective structure group action $\triangleright\colon G \times F \to F$ between $\pi\colon E_1 \to M$ and $\pi\colon E_2 \to M$, provided the existence of the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$.

Subproof (Constructing an isomorphism of fibre bundles with effective structure group action). For each $i \in \mathcal{I}$, we define a smooth map given by

$$\Phi_i\colon {\pi_1}^{-1}[U_i] \to {\pi_2}^{-1}[U_i], \quad a \mapsto (\pi_2, \alpha_i^2)^{-1}\left(\pi_1(a), \nu_i(\pi_1(a)) \triangleright \alpha_i^1(a)\right). \tag{3.6}$$

This map takes a point a in the fibre ${\pi_1}^{-1}[\{p\}]$ of E_1 over a point $p \in U_i$ and assigns to it a value in the typical fibre F by means of the bundle chart $\alpha_i^1 \in \mathcal{B}_1$. We then act on that value by the group element

$\nu_i(\pi_1(p))$ before using the bundle chart $\alpha_i^2 \in \mathcal{B}_2$ in order to obtain a point $\Phi_i(a)$ in the fibre ${\pi_2}^{-1}[\{p\}]$ of E_2. We claim that this property ensures that Φ_i and Φ_j agree on their domain intersection ${\pi_1}^{-1}[U_i \cap U_j]$ for all $i, j \in \mathcal{I}$. In order to see this, suppose α_i and α_j are two bundle charts of $\mathcal{B}_1$ with non-empty domain intersection $\pi_1^{-1}[U_i \cap U_j]$. The claim is proven if we can show that for any point $a \in {\pi_1}^{-1}[U_i \cap U_j]$ it holds that

$$\left(\alpha_j^2 \circ \Phi_i\right)(a) = \nu_j(\pi_1(a)) \triangleright \alpha_j^1(a). \tag{3.7}$$

Using the definition of Φ_i, the left hand side becomes

$$\left(\alpha_j^2 \circ \Phi_i\right)(a) = \left(\alpha_j^2 \circ (\pi_2, \alpha_i^2)^{-1}\right)\left(\pi_1(a), \nu_i\left(\pi_1(a)\right) \triangleright \alpha_i^1(a)\right),$$

which by means of the transition function $\rho_{ji}^2 \colon U_i \cap U_j \to G$ is readily expressed as

$$= \rho_{ji}^2(\pi_1(a)) \triangleright \left(\nu_i\left(\pi_1(a)\right) \triangleright \alpha_i^1(a)\right)$$

and equal to

$$= \left(\rho_{ji}^2(\pi_1(a)) \bullet \nu_i\left(\pi_1(a)\right)\right) \triangleright \alpha_i^1(a)$$

since $\triangleright \colon G \times F$ is a left action. The hypothesis of the proposition, eq. (3.5), allows us to write

$$= \left(\nu_j\left(\pi_1(a)\right) \bullet \rho_{ji}^1(\pi_1(a))\right) \triangleright \alpha_i^1(a),$$

which is equal to

$$= \nu_j\left(\pi_1(a)\right) \triangleright \left(\rho_{ji}^1\left(\pi_1(a)\right) \triangleright \alpha_i^1(a)\right),$$

$\triangleright \colon G \times F \to F$ being a left action. Letting the transition function $\rho_{ji}^1 \colon U_i \cap U_j \to G$ act on $\alpha_i^1(a)$ yields precisely the right hand side of eq. (3.7):

$$= \nu_j(\pi_1(a)) \triangleright \alpha_j^1(a).$$

This proves that the map

$$\Phi \colon E_1 \to E_2, \quad a \mapsto \left(\pi_2, \alpha_i^2\right)^{-1}\left(\pi_1(a), \nu_i\left(\pi_1(a)\right) \triangleright \alpha_i^1(a)\right) \text{ for } i \in \mathcal{I} \colon \pi_1(a) \in U_i \tag{3.8}$$

is a well-defined smooth map. Under exchange of $\nu_i \colon U_i \to G$ with $\tilde{\nu}_i \colon U_i \to G, p \mapsto \nu_i(p)^{-1}$, the above reasoning proves that the map

$$\Psi \colon E_2 \to E_1, \quad b \mapsto \left(\pi_1, \alpha_i^1\right)^{-1}\left(\pi_2(b), \nu_i\left(\pi_2(b)\right)^{-1} \triangleright \alpha_i^2(b)\right) \text{ for } i \in \mathcal{I} \colon \pi_2(b) \in U_i \tag{3.9}$$

is well-defined and smooth as well. From eqs. (3.8) and (3.9) we read off that $\pi_2 \circ \Phi = \pi_1$ and $\pi_1 \circ \Psi = \pi_2$, i.e., both $\Phi \colon E_1 \to E_2$ and $\Psi \colon E_2 \to E_2$ preserve fibres. In order to show that $\Phi \colon E_1 \to E_2$ is a bundle isomorphism along the identity morphism id_M, it is left to show that $\Psi \colon E_2 \to E_1$ is the inverse of $\Phi \colon E_1 \to E_2$. To this end, pick an arbitrary point $a \in E_1$ and fix $i \in \mathcal{I}$ such that $\pi_1(a) \in U_i$. Using the definitions of both Φ and Ψ we discover that

$$\begin{aligned}
\left(\Psi \circ \Phi\right)(a) &= \left(\pi_1, \alpha_i^1\right)^{-1}\left(\pi_1(a), \nu_i\left(\pi_2(\Phi(a))\right)^{-1} \triangleright \alpha_i^2\left(\Phi(a)\right)\right) \\
&= \left(\pi_1, \alpha_i^1\right)^{-1}\left(\pi_1(a), \nu_i\left(\pi_1(a)\right)^{-1} \triangleright \left(\nu_i(\pi_1(a)) \triangleright \alpha_i^1(a)\right)\right).
\end{aligned}$$

It is straightforward to show that the left action $\triangleright \colon G \times F \to F$ satisfies $g^{-1} \triangleright (g \triangleright \xi) = \xi$ for all $g \in G$ and $\xi \in F$. Consequently,

$$\begin{aligned}
&= \left(\pi_1, \alpha_i^1\right)^{-1}\left(\pi_1(a), \alpha_i^1(a)\right), \\
&= a.
\end{aligned}$$

Again, by symmetry, it also follows that $\Phi \circ \Psi = \mathrm{id}_{E_2}$. It remains to check that $\Phi\colon E_1 \to E_2$ is an isomorphism of fibre bundles over M with effective structure group action $\triangleright\colon G \times F \to F$. In fact, eq. (3.7) of this proof already verified that eq. (3.4) holds for the smooth map $\nu_i\colon U_i \to G$ when $\alpha_i^1 \in \mathcal{B}_2$ and $\alpha_i^2 \in \mathcal{B}_2$ are chosen. Let us verify this result for arbitrary bundle charts $\alpha_i^1 \in \mathcal{B}_1$ and $\alpha_j^2 \in \mathcal{B}_2$ with non-empty domain intersection $U_i \cap U_j \neq \emptyset$. Let $p \in U_i \cap U_j$ and $\xi \in F$, then

$$\left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi) = \left(\alpha_j^2 \circ \left(\pi_2, \alpha_i^2\right)^{-1}\right)(p, \nu_i(p) \triangleright \xi)$$

using the definition of $\Phi\colon E_1 \to E_2$, and

$$= \rho_{ji}^2(p) \triangleright (\nu_i(p) \triangleright \xi)$$

using the transition function $\rho_{ji}^2\colon U_i \cap U_j \to G$. Finally, since $\triangleright\colon G \times F \to F$ is a left action, we obtain

$$= \left(\rho_{ji}^2(p) \bullet \nu_i(p)\right) \triangleright \xi.$$

With that we found a smooth map $\tau_{ji}\colon U_i \cap U_j \to G, p \mapsto \rho_{ji}^2(p) \bullet \nu_i(p)$ satisfying eq. (3.4) for arbitrary bundle charts $\alpha_i^1 \in \mathcal{B}_1$ and $\alpha_j^2 \in \mathcal{B}_2$. This concludes the first part of the proof.

Let us now prove the second part of the proposition. Suppose we are given an isomorphism $\Phi\colon E_1 \to E_2$ of fibre bundles over M with effective structure group action $\triangleright\colon G \times F \to F$ between $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ with respect to their respective $(G, \triangleright)$-bundle structures $[\mathcal{B}_1]_{(G,\triangleright)}$ and $[\mathcal{B}_2]_{(G,\triangleright)}$.

Subproof (Retrieving the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$). It is fairly straightforward to recover the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$; they are precisely a subset of the maps appearing in the definition of what it means for $\Phi\colon E_1 \to E_2$ to be an isomorphism of fibre bundles over M with effective structure group action $\triangleright\colon G \times F \to F$. Let $i \in \mathcal{I}$, then $\alpha_i^1 \in \mathcal{B}_1$ and $\alpha_i^2 \in \mathcal{B}_2$. Consequently, by definition 3.6, there exists a smooth map $\nu_i\colon U_i \to G$ such that

$$\forall p \in U_i\colon \forall \xi \in F\colon \quad \alpha_i^2 \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}(p, \xi) = \nu_i(p) \triangleright \xi. \tag{3.10}$$

It remains to check that the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$ correctly relates the transition functions $\left\{\rho_{ji}^1\colon U_i \cap U_j \to G \mid i, j \in \mathcal{I}\right\}$ of $\mathcal{B}_1$ to the transition functions $\left\{\rho_{ji}^2\colon U_i \cap U_j \to G \mid i, j \in \mathcal{I}\right\}$ of $\mathcal{B}_2$. Let $i, j \in \mathcal{I}$ such that $U_i \cap U_j \neq \emptyset$ and consider the expression

$$\left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi), \tag{3.11}$$

where $p \in U_i \cap U_j$ and $\xi \in F$. On the one hand, we can insert the identity $\left(\pi_2, \alpha_i^2\right)^{-1} \circ \left(\pi_2, \alpha_i^2\right)$ in between α_j^2 and Φ in order to obtain

$$\begin{aligned}
\left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi) &= \left(\alpha_j^2 \circ \left(\pi_2, \alpha_i^2\right)^{-1} \circ \left(\pi_2, \alpha_i^2\right) \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi) \\
&= \left(\alpha_j^2 \circ \left(\pi_2, \alpha_i^2\right)^{-1}\right)(p, \nu_i(p) \triangleright \xi) \\
&= \rho_{ji}^2(p) \triangleright (\nu_i(p) \triangleright \xi).
\end{aligned}$$

On the other hand, we can insert the identity $\left(\pi_1, \alpha_j^1\right)^{-1} \circ \left(\pi_1, \alpha_j^1\right)$ in between Φ and $\left(\pi_1, \alpha_i^1\right)^{-1}$. Then:

$$\begin{aligned}
\left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi) &= \left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_j^1\right)^{-1} \circ \left(\pi_1, \alpha_j^1\right) \circ \left(\pi_1, \alpha_i^1\right)^{-1}\right)(p, \xi) \\
&= \left(\alpha_j^2 \circ \Phi \circ \left(\pi_1, \alpha_j^1\right)^{-1}\right)\left(p, \rho_{ji}^1(p) \triangleright \xi\right) \\
&= \nu_j(p) \triangleright \left(\rho_{ji}^1(p) \triangleright \xi\right).
\end{aligned}$$

Since $\triangleright\colon G \to F$ is a left action and $\xi \in F$ arbitrary, it follows by comparison that $\rho_{ji}^2(p) \bullet \nu_i(p) = \nu_j(p) \bullet \rho_{ji}^1(p)$ holds for all $p \in U_i \cap U_j$. This concludes the proof.

Remark 3.4. Note that by an appropriate choice of either $\mathcal{B}_1$ or $\mathcal{B}_2$ from the respective $(G, \triangleright)$-bundle structures of isomorphic fibre bundles $\pi_1 \colon E_1 \to M$ and $\pi_2 \colon E_2 \to M$ with effective structure group action $\triangleright \colon G \times F \to F$, we can always make their G-valued transition functions coincide, such that $\nu_i \colon U_i \to G, p \mapsto e_G$ for all $i \in \mathcal{I}$.

Theorem 3.3 (Fibre bundle construction theorem)**.** *Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ and $(F, \mathcal{O}_F, \mathcal{A}_F)$ be non-empty smooth manifolds, $(G, \mathcal{O}_G, \mathcal{A}_G, \bullet)$ a Lie group and $\triangleright \colon G \times F \to F$ an effective Lie group left action.*

Suppose we are given $\mathcal{I}$ such that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M and $\{\rho_{ij} \colon U_i \cap U_j \to G \mid i, j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\}$ is a collection of smooth maps satisfying the following cocycle conditions:

$$\forall i \in \mathcal{I} \colon \forall p \in U_i \colon \qquad \rho_{ii}(p) = e_G, \tag{3.12}$$

$$\forall i, j, k \in \mathcal{I} \colon \ [U_i \cap U_j \cap U_k \neq \emptyset \implies \forall p \in U_i \cap U_j \cap U_k \colon \qquad \rho_{ik}(p) = \rho_{ij}(p) \bullet \rho_{jk}(p)] \,. \tag{3.13}$$

Then there exists a fibre bundle $\pi \colon E \to M$ over M with typical fibre F and a $(G, \triangleright)$-compatible bundle atlas $\mathcal{B} = \{\alpha_i \colon \pi^{-1}[U_i] \to F \mid i \in \mathcal{I}\}$ whose transition functions agree with the set

$$\{\rho_{ij} \triangleright \colon U_i \cap U_j \to \mathrm{Diff}(F), p \mapsto \rho_{ij}(p) \triangleright \mid i, j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\} \,. \tag{3.14}$$

In particular, $\pi \colon E \to M$ together with the $(G, \triangleright)$-bundle structure $[\mathcal{B}]_{(G, \triangleright)}$ qualifies as a fibre bundle with effective structure group action.

Moreover, by proposition 3.2, $\pi \colon E \to M$ is unique up to isomorphism of fibre bundles with effective structure group action $\triangleright \colon G \times F \to F$.

Proof 3.3. The conceptual idea is to consider for any $i \in \mathcal{I}$ the product manifold $U_i \times F$ as a patch of the fibre bundle to be constructed over which the fibre bundle is trivial. Given two such patches $U_i \times F$ and $U_j \times F$ with overlap in the sense of $U_i \cap U_j \neq \emptyset$, we regard them as disjoint sets which we glue together by means of the transition function $\rho_{ij} \triangleright$.

We formalize this idea in the following way:

We start off with the disjoint union of the patches $U_i \times F$

$$\mathcal{E} := \bigcup \{U_i \times F \times \{i\} \mid i \in \mathcal{I}\} \subseteq M \times F \times \mathcal{I} \tag{3.15}$$

as underlying set on which we establish an equivalence relation $\sim$ according to

$$\forall (p, \phi, i), (q, \psi, j) \in \mathcal{E} \colon \ [(p, \phi, i) \sim (q, \psi, j) :\iff p = q \land \phi = \rho_{ij}(p) \triangleright \psi] \,. \tag{3.16}$$

It is an equivalence relation due to the cocycle conditions that the collection of smooth maps

$$\{\rho_{ij} \colon U_i \cap U_j \to G \mid i, j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\} \tag{3.17}$$

satisfies. As we see below, this correctly glues the different patches together.

The set that we equip with the structure of a fibre bundle with the given effective structure group action $\triangleright \colon G \times F \to F$ is the set of equivalence classes $E := \mathcal{E}/_{\sim}$.

Note that the projection onto the first argument $\mathrm{pr}_1 \colon \mathcal{E} \to M$ is constant on the equivalence classes. It thus induces a map $\pi \colon E \to M$, the projection of the fibre bundle to be constructed.

Subproof ($\pi \colon E \to M$ is a surjection). The projection $\pi \colon E \to M$ is a surjection since $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M. For, if p is a point in M, then there exists $i \in \mathcal{I}$ such that $p \in U_i$. By hypothesis F is non-empty. Hence there exists $\xi \in F$, and consequently, we have an element $[(p, \xi, i)]_{\sim} \in E$ that satisfies $\pi([(p, \xi, i)]_{\sim}) = \mathrm{pr}_1((p, \xi, i)) = p$.

Subproof (Providing a proto bundle atlas for $\pi\colon E \to M$). We suggest the following collection of maps as a prototype of the bundle atlas to be constructed:

$$\mathcal{B} := \left\{\alpha_i\colon \pi^{-1}[U_i] \to F,\ [(p,\phi,i)]_\sim \mapsto \phi \mid i \in \mathcal{I}\right\}. \tag{3.18}$$

Note that for each $i \in \mathcal{I}$, the map $\alpha_i\colon \pi^{-1}[U_i] \to F$ is indeed well-defined. Firstly, every point $a \in \pi^{-1}[U_i]$ can be represented by (p,ϕ,i) for some $p \in U_i$ and $\phi \in F$. Secondly, the element $\phi \in F$ with this property is unique. For, if there exists another $\phi' \in F$ such that $a = [(p,\phi',i)]_\sim$ then $(p,\phi',i) \sim (p,\phi,i)$, which by definition means that $\phi' = \rho_{ii}(p) \triangleright \phi$. However, since $\rho_{ii}(p) = e$ by hypothesis and due to the fact that $\triangleright\colon G \times F \to F$ is a left action, it follows that $\phi' = e \triangleright \phi = \phi$. We conclude that $\alpha_i\colon \pi^{-1}[U_i] \to F$ is well-defined.

We will now show that $\mathcal{B}$ satisfies the hypotheses (1)-(4) from Theorem 2.1 (Equipping a set with the structure of a fibre bundle using a bundle atlas).

Let $i \in \mathcal{I}$. By definition, we have that $\mathrm{Dom}(\alpha_i) = \pi^{-1}[U_i]$. Since $\pi\colon E \to M$ is a surjection, it holds that $U_i = \pi[\mathrm{Dom}(\alpha_i)]$. This proves hypothesis (2), i.e., $\mathrm{Dom}(\alpha_i) = \pi^{-1}[\pi[\mathrm{Dom}(\alpha_i)]]$. It also proves that $\{\pi[\mathrm{Dom}(\alpha_i)] \mid i \in \mathcal{I}\}$ is an open cover of M, being that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M by assumption. This proves hypothesis (1). Now consider the map

$$(\pi,\alpha_i)\colon \pi^{-1}[U_i] \to U_i \times F, \quad [(p,\phi,i)]_\sim \mapsto (p,\phi). \tag{3.19}$$

It is well-defined as both $\pi\colon E \to M$ and $\alpha_i\colon \pi^{-1}[U_i] \to F$ are well-defined. It is readily verified that it is bijective with inverse

$$(\pi,\alpha_i)^{-1}\colon U_i \times F \to \pi^{-1}[U_i], \quad (p,\phi) \mapsto [(p,\phi,i)]_\sim. \tag{3.20}$$

This proves hypothesis (3). It remains to check hypothesis (4). To this end let $i,j \in \mathcal{I}$ such that $U_i \cap U_j \neq \emptyset$ and let $a \in \pi^{-1}[U_i \cap U_j]$. We have seen earlier that there exist $p \in U_i \cap U_j$ and $\phi,\psi \in F$ such that $a = [(p,\phi,i)]_\sim = [(p,\psi,j)]_\sim$. By definition of $\sim$, the relationship $\psi = \rho_{ji}(p) \triangleright \phi$ holds. This relationship determines the map

$$(\pi,\alpha_j) \circ (\pi,\alpha_i)^{-1}\colon\ (U_i \cap U_j) \times F \to (U_i \cap U_j) \times F, \quad (p,\phi) \mapsto (p, \rho_{ji}(p) \triangleright \phi). \tag{3.21}$$

Written in this way, it is evidently identified as a smooth map, being a tuple of smooth maps.

Subproof ($\mathcal{B}$ is $(G,\triangleright)$-compatible). It remains to check that $\mathcal{B}$ is a $(G,\triangleright)$-compatible bundle atlas with transition functions provided by the set $\{\rho_{ij}\triangleright\colon U_i \cap U_j \to \mathrm{Diff}(F), p \mapsto \rho_{ij}(p)\triangleright \mid i,j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$. This is readily verified by inspection of equation (3.21).

3.1 Vector bundles

From the point of view of a physicist, we desire to make contact with the actual structure that we want to model. That is why we pursue a rather verbose definition of what a *vector bundle* encompasses.

Definition 3.7 (Vector bundle (of rank k)). Let $\pi\colon E \to M$ be a fibre bundle over M with typical fibre V. $\pi\colon E \to M$ is said to be a **vector bundle of rank k over M with typical fibre V** if:

1. V is equipped with a k-dimensional real vector space structure compatible with its smooth structure (in the sense that any linear isomorphism $\varphi\colon V \to \mathbb{R}^k$ is a diffeomorphism),
2. for any point $p \in M$, the fibre $E_p = \pi^{-1}[\{p\}]$ is equipped with the structure of a k-dimensional real vector space,
3. there exists a bundle atlas $\mathcal{B}$ consisting of (fibre-wisely) **linear bundle charts** (also referred to as **vector bundle charts**), i.e.,

$$\forall \alpha \in : \forall p \in \pi[\mathrm{Dom}(\alpha)]\colon\ \left(\alpha|_{E_p}\colon E_p \to V \text{ is a linear isomorphism}\right). \tag{3.22}$$

Example 3.1. The tangent bundle $\pi\colon TM \to M$ of a smooth manifold M is a vector bundle.

Proposition 3.4 (Vector bundle is fibre bundle with general linear structure group action)**.** *A vector bundle $\pi\colon E \to M$ with typical fibre V and vector bundle atlas $\mathcal{B}$ can be canonically regarded as a fibre bundle with effective structure group action with respect to the defining representation of* $\mathrm{GL}(k,\mathbb{R})$*, and vice-versa.*

Proof 3.4. We first pick a linear diffeomorphism $\varphi\colon V \to \mathbb{R}^k$. Its existence is guaranteed by item 1 from definition 3.7. We then can define $\tilde{\mathcal{B}} := \{\varphi \circ \alpha \mid \alpha \in \mathcal{B}\}$. It holds that $\pi\colon E \to M$ is a vector bundle with typical fibre $\mathbb{R}^k$ and vector bundle atlas $\tilde{\mathcal{B}}$. Let us take a look at the transition functions of $\tilde{\mathcal{B}}$. For any two vector bundle charts $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{B}}$, we have

$$\rho_{\tilde{\beta}\tilde{\alpha}}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \to \mathrm{Diff}(\mathbb{R}^k), \quad p \mapsto \varphi \circ \beta|_{E_p} \circ \alpha|_{E_p}^{-1} \circ \varphi^{-1}. \tag{3.23}$$

In fact, $\rho_{\tilde{\beta}\tilde{\alpha}}$ maps into $\mathrm{GL}(k,\mathbb{R})$, being that its value at p is the composition of linear isomorphisms. Moreover, it smoothly depends on p as it is the composition of smooth maps. This becomes apparent when we express $\rho_{\tilde{\beta}\tilde{\alpha}}(p)$ explicitly and in components

$$\Big(\rho_{\tilde{\beta}\tilde{\alpha}}(p)\Big)^n{}_m = \varphi^n \circ \beta \circ (\pi,\alpha)^{-1}\left(p, \varphi^{-1}(e_m)\right) \qquad \text{for } 1 \le m, n \le k, \tag{3.24}$$

where e_m is the m-th element of the standard basis of $\mathbb{R}^k$. This proves the first part of the claim.

The converse direction is verified in straightforward fashion. Given a fibre bundle $\pi\colon E \to M$ with typical fibre $\mathbb{R}^k$ and $(\mathrm{GL}(k,\mathbb{R}),)$-compatible bundle atlas $\mathcal{B}$, all we have to do is equip $\mathbb{R}^k$ with its canonical k-dimensional real vector space structure and subsequently transfer it onto each fibre E_p by means of the bundle charts of $\mathcal{B}$. This is done consistently due to the fact that $\mathcal{B}$ is $(\mathrm{GL}(k,\mathbb{R}),)$-compatible.

Definition 3.8 (Vector bundle morphism)**.** Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be vector bundles. A bundle morphism (Φ,φ) from $\pi\colon E \to M$ to $\pi'\colon E' \to M'$ is said to be a **vector bundle morphism** if for every $p \in M$ the restriction $\Phi|_{E_p}\colon E_p \to E'_{\varphi(p)}$ is a linear map.

Definition 3.9 (Vector bundle isomorphism)**.** A vector bundle morphism (Φ,φ) that is also a bundle isomorphism is said to be a **vector bundle isomorphism**. This is precisely the case if (Φ^{-1},φ^{-1}) is defined and a vector bundle morphism as well.

Remark 3.5 (Every vector bundle admits global section)*.* Note that a vector bundle $\pi\colon E \to M$ over M of rank k admits a global section. One such section is the null section $0_{\Gamma(\pi)}\colon M \to E, p \mapsto 0_{E_p}$.

Definition 3.10. The **pointwise addition of sections** of a vector bundle is the operation

$$\underset{\Gamma(E)}{\oplus}\colon \Gamma(E) \times \Gamma(E) \to \Gamma(E), \quad (Y,Z) \mapsto Y \underset{\Gamma(E)}{\oplus} Z, \tag{3.25}$$

where

$$Y \underset{\Gamma(E)}{\oplus} Z\colon M \to E, \quad p \mapsto Y(p) \underset{E_p}{\oplus} Z(p). \tag{3.26}$$

Definition 3.11. The $C^\infty(M)$**-scalar multiplication**, or **pointwise scalar multiplication**, on the set of sections of a vector bundle is the operation

$$\underset{\Gamma(E)}{\boxdot}\colon C^\infty(M) \times \Gamma(E) \to \Gamma(E), \quad (\varphi, Y) \mapsto \varphi \underset{\Gamma(E)}{\boxdot} Y, \tag{3.27}$$

where

$$\varphi \underset{\Gamma(E)}{\boxdot} Y\colon M \to E, \quad p \mapsto \varphi(p) \underset{E_p}{\boxdot} Y(p). \tag{3.28}$$

Exercise 3.1. Verify that the set of sections $\Gamma(E)$ of a vector bundle, together with pointwise addition $\oplus_{\Gamma(E)}$ and pointwise scalar multiplication $\boxdot_{\Gamma(E)}$ forms a unital module $(\Gamma(E), \oplus_{\Gamma(E)}, \boxdot_{\Gamma(E)})$ over the unital commutative ring $(C^\infty(M), +_{C^\infty(M)}, \cdot_{C^\infty(M)})$ of the smooth functions on M.

3.2 Affine Bundles

There exists a noteworthy generalization of vector bundles with relevance for teleparallel gravity, the class of *affine bundles*. Refer to appendix D for a short review of affine spaces.

Definition 3.12 (Affine bundle (of rank k))**.** Let $\pi\colon E \to M$ be a fibre bundle over M with typical fibre A. $\pi\colon E \to M$ is said to be an **affine bundle of rank k over M with typical fibre A** if:

1. A is equipped with a k-dimensional real affine space structure $(A, V, \boxplus)$ compatible with its smooth structure (in the sense that any affine isomorphism $\phi\colon A \to \mathbb{R}^k$ is a diffeomorphism)
2. for any point $p \in M$, the fibre E_p is equipped with the structure of a k-dimensional real affine space $(E_p, \bar{E}_p, \boxplus_p)$
3. there exists a bundle atlas $\mathcal{B}$ of $\pi\colon E \to M$ consisting of fibre-wisely **affine bundle charts**, i.e.,

$$\forall \alpha \in \mathcal{B}\colon \forall p \in \pi\left[\mathrm{Dom}(\alpha)\right]: \left(\alpha|_{E_p}\colon E_p \to A \text{ is an affine isomorphism}\right). \tag{3.29}$$

Example 3.2. Every vector bundle can be canonically understood as an affine bundle due to the fact that a vector space can be understood as an affine space over itself in a canonical way.

Proposition 3.5 (Affine bundle is fibre bundle with general affine structure group action)**.** *An affine bundle $\pi\colon E \to M$ with typical fibre A and affine bundle atlas $\mathcal{B}$ can be canonically regarded as a fibre bundle with effective structure group action with respect to the defining representation of* $\mathrm{GA}(k, \mathbb{R})$*, and vice-versa.*

Proof 3.5. We first pick an affine diffeomorphism $\phi\colon A \to \mathbb{R}^k$. Its existence is guaranteed by item 1 from definition 3.12. We then can define $\tilde{\mathcal{B}} \coloneqq \{\phi \circ \alpha \mid \alpha \in \mathcal{B}\}$. It holds that $\pi\colon E \to M$ is an affine bundle with typical fibre $\mathbb{R}^k$ and affine bundle atlas $\tilde{\mathcal{B}}$. Let us take a look at the transition functions of $\tilde{\mathcal{B}}$. For any two affine bundle charts $\tilde{\alpha}, \tilde{\beta} \in \tilde{\mathcal{B}}$, we have

$$\rho_{\tilde{\beta}\tilde{\alpha}}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \to \mathrm{Diff}(\mathbb{R}^k), \quad p \mapsto \phi \circ \beta|_{E_p} \circ \alpha|_{E_p}^{-1} \circ \phi^{-1}. \tag{3.30}$$

In fact, $\rho_{\tilde{\beta}\tilde{\alpha}}$ maps into $\mathrm{GA}(k, \mathbb{R})$, being that its value at p is the composition of affine isomorphisms. Moreover, it smoothly depends on p as it is the composition of smooth maps. This becomes apparent when we express $\rho_{\tilde{\beta}\tilde{\alpha}}(p)$ explicitly and in components

$$\left(\rho_{\tilde{\beta}\tilde{\alpha}}(p)(0)\right)^n = \phi^n \circ \beta \circ (\pi, \alpha)^{-1}\left(p, \phi^{-1}(0)\right) \qquad \text{for } 1 \leq n \leq k, \tag{3.31}$$

$$\left(\overrightarrow{\rho_{\tilde{\beta}\tilde{\alpha}}(p)}\right)^n_{\ m} = \phi^n \circ \beta \circ (\pi, \alpha)^{-1}\left(p, \phi^{-1}(e_m)\right) - \left(\rho_{\tilde{\beta}\tilde{\alpha}}(p)(0)\right)^n \qquad \text{for } 1 \leq m, n \leq k, \tag{3.32}$$

where e_m is the m-th element of the standard basis of $\mathbb{R}^k$. This proves the first part of the claim.

The converse direction is verified in straightforward fashion. Given a fibre bundle $\pi\colon E \to M$ with typical fibre $\mathbb{R}^k$ and $(\mathrm{GA}(k, \mathbb{R}),)$-compatible bundle atlas $\mathcal{B}$, all we have to do is equip $\mathbb{R}^k$ with its canonical k-dimensional real affine structure and subsequently transfer it onto each fibre E_p by means of the bundle charts of $\mathcal{B}$. This is done consistently due to the fact that $\mathcal{B}$ is $(\mathrm{GA}(k, \mathbb{R}),)$-compatible.

Definition 3.13 (Affine bundle morphism)**.** Let $\pi\colon E \to M$ and $\pi'\colon E' \to M'$ be affine bundles. A bundle morphism (Φ, φ) from $\pi\colon E \to M$ to $\pi'\colon E' \to M'$ is said to be an **affine bundle morphism** if for every $p \in M$ the restriction $\Phi|_{E_p}\colon E_p \to E'_{\varphi(p)}$ is an affine map.

Definition 3.14 (Affine bundle isomorphism)**.** An affine bundle morphism (Φ, φ) that is also a bundle isomorphism is said to be a **affine bundle isomorphism**. This is precisely the case if $(\Phi^{-1}, \varphi^{-1})$ is defined and an affine bundle morphism as well.

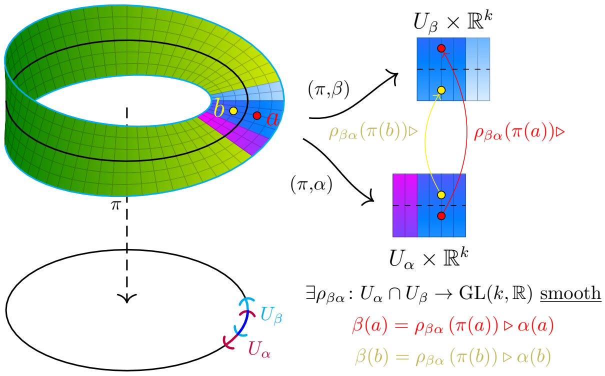

Figure 3.2: The Möbius strip may be given a $\mathrm{GL}(1,\mathbb{R})$-compatible bundle atlas that qualifies it as a vector bundle.

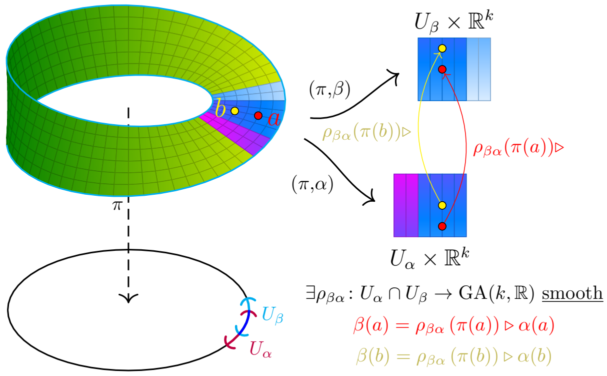

Figure 3.3: Alternatively, the Möbius strip may be given a $\mathrm{GA}(1,\mathbb{R})$-compatible bundle atlas that renders it into an affine bundle.

Definition 3.15 (Affine bundle modelled on a vector bundle)**.** Let $\vec{\pi}\colon \vec{E} \to M$ be a vector bundle with typical fibre V. An **affine bundle modelled on** $\vec{\pi}\colon \vec{E} \to M$ is a fibre bundle $\pi\colon E \to M$ over the same base manifold M whose typical fibre A is an affine space $(A, V, \boxplus)$ modelled on V with the following properties:

1. A is equipped with a k-dimensional real affine space structure $(A, V, \boxplus)$ compatible with its smooth structure (in the sense that any linear isomorphism $\varphi\colon V \to \mathbb{R}^k$ and any affine isomorphism $\phi\colon A \to \mathbb{R}^k$ are diffeomorphisms, thus also rendering $\boxplus\colon A \times V \to A$ and $\boxminus\colon A \times A \to V$ smooth)
2. for any point $p \in M$, the fibre E_p is equipped with the structure of an affine space $(E_p, \vec{E}_p, \boxplus_p)$ modelled on the fibre $\vec{E}_p$ of the vector bundle
3. there exist a bundle atlas $\mathcal{B}$ of $\pi\colon E \to M$ consisting of fibre-wisely **affine bundle charts**, i.e.,

$$\forall \alpha \in \mathcal{B}\colon \forall p \in \pi\left[\mathrm{Dom}(\alpha)\right] : \left(\alpha|_{E_p}\colon E_p \to A \text{ is an affine isomorphism}\right), \tag{3.33}$$

and such that for any bundle chart $\alpha \in \mathcal{B}$ there exists a vector bundle chart $\vec{\alpha}$ of $\vec{\pi}\colon \vec{E} \to M$ with $\vec{\pi}\left[\mathrm{Dom}\left(\vec{\alpha}\right)\right] = \pi[\mathrm{Dom}(\alpha)]$ such that

$$\forall p \in \pi[\mathrm{Dom}(\alpha)]\colon \overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}. \tag{3.34}$$

Lemma 3.6 (Affine bundle chart induces vector bundle chart)**.** *Let $\pi\colon E \to M$ be an affine bundle modelled on $\vec{\pi}\colon \vec{E} \to M$. For every affine bundle chart α of $\pi\colon E \to M$ there exists a vector bundle chart $\vec{\alpha}$ of $\vec{\pi}\colon \vec{E} \to M$ with $\vec{\pi}\left[\mathrm{Dom}(\vec{\alpha})\right] = \pi[\mathrm{Dom}(\alpha)]$ such that for every point $p \in \pi[\mathrm{Dom}(\alpha)]$ it holds that*

$$\overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}. \tag{3.35}$$

Proof 3.6. The obvious candidate for the wanted vector bundle chart is given by

$$\vec{\alpha}\colon \vec{\pi}^{-1}\left[\pi[\mathrm{Dom}(\alpha)]\right] \to V, \quad \vec{a} \mapsto \overrightarrow{\alpha|_{E_{\vec{\pi}(\vec{a})}}}(\vec{a}). \tag{3.36}$$

We have to show that it is a vector bundle chart. First, note that for any $p \in \pi[\mathrm{Dom}(\alpha)]$ the restriction $\vec{\alpha}|_{\vec{E}_p} = \overrightarrow{\alpha|_{E_p}}\colon \vec{E}_p \to V$ is a linear isomorphism, being that $\alpha|_{E_p}\colon E_p \to A$ is an affine isomorphism. It is left to show that

$$(\vec{\pi}, \vec{\alpha})\colon \vec{\pi}^{-1}\left[\pi[\mathrm{Dom}(\alpha)]\right] \to \pi[\mathrm{Dom}(\alpha)] \times V \tag{3.37}$$

is a diffeomorphism. Let $p \in \pi[\mathrm{Dom}(\alpha)]$ and let β be an affine bundle chart of $\pi\colon E \to M$ with the property that there exists a vector bundle chart of $\vec{\pi}\colon \vec{E} \to M$ with $p \in \vec{\pi}[\mathrm{Dom}(\vec{\beta})] = \pi[\mathrm{Dom}(\beta)]$ and such that for all $q \in \pi[\mathrm{Dom}(\beta)]$ it holds that $\vec{\beta}|_{\vec{E}_q} = \overrightarrow{\beta|_{E_q}}$, whose existence is guaranteed by definition 3.15. We want to show that

$$\vec{\alpha} \circ (\vec{\pi}, \vec{\beta})^{-1}(q, v) = \overrightarrow{\rho_{\alpha\beta}(q)}(v). \tag{3.38}$$

depends smoothly on $q \in \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)]$ and $v \in V$. To this end, fix an affine diffeomorphism $\phi\colon A \to \mathbb{R}^k$. Note that $\vec{\phi}\colon V \to \mathbb{R}^k$ is a linear diffeomorphism. In proof of proposition 3.5 we have seen that

$$\rho_{\tilde{\alpha}\tilde{\beta}}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \to \mathrm{GA}(k, \mathbb{R}), \quad q \mapsto \phi \circ \alpha|_{E_q} \circ \beta|_{E_q}^{-1} \circ \phi^{-1} \tag{3.39}$$

and thus also

$$\vec{\rho}_{\tilde{\alpha}\tilde{\beta}}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \to \mathrm{GL}(k, \mathbb{R}), \quad q \mapsto \vec{\phi} \circ \overrightarrow{\alpha|_{\vec{E}_q}} \circ \overrightarrow{\beta|_{\vec{E}_q}}^{-1} \circ \vec{\phi}^{-1} \tag{3.40}$$

are smooth. Consequently, the expression

$$\vec{\phi} \circ \vec{\alpha} \circ (\vec{\pi}, \vec{\beta})^{-1}(q, \vec{\phi}^{-1}(\xi)) = \vec{\rho}_{\tilde{\alpha}\tilde{\beta}}(q)(\xi) \tag{3.41}$$

depends smoothly on q and $\xi \in \mathbb{R}^k$. Since $\vec{\phi}\colon V \to \mathbb{R}$ is a global chart of V, smoothness of (3.38) is proven. By symmetry, also $\vec{\beta} \circ (\vec{\pi}, \vec{\alpha})^{-1}(q, v)$ depends smoothly on q and v. This concludes the proof.

Remark 3.6 (Every affine bundle modelled on a vector bundle is an affine bundle). It is immediate that an affine bundle $\pi\colon E \to M$ modelled on a vector bundle $\vec{\pi}\colon \vec{E} \to M$ is an affine bundle, thus justifying the terminology. The converse holds true as well. Every affine bundle $\pi\colon E \to M$ is modelled on some vector bundle $\vec{\pi}\colon \vec{E} \to M$. This is the subject of the theorem 3.7.

Theorem 3.7 (Every affine bundle is modelled on a vector bundle). *For any affine bundle $\pi\colon E \to M$ there exists a vector bundle $\vec{\pi}\colon \vec{E} \to M$ such that $\pi\colon E \to M$ is modelled on $\vec{\pi}\colon \vec{E} \to M$. Moreover, the vector bundle $\vec{\pi}\colon \vec{E} \to M$ is unique up to vector bundle isomorphism.*

Proof 3.7. Suppose we are given an affine bundle $\pi\colon E \to M$ of rank k with affine bundle atlas $\mathcal{B} = \{\alpha_i\colon \pi^{-1}[U_i] \to A \mid i \in \mathcal{I}\}$. The proof naturally splits into two parts: the existence of a vector bundle $\vec{\pi}\colon \vec{E} \to M$ such that $\pi\colon E \to M$ is modelled on $\vec{\pi}\colon \vec{E} \to M$ and its uniqueness up to vector bundle isomorphism.

Subproof (Existence). For simplicity, let us assume that the fibre $A = \mathbb{R}^k$ is chosen, cf. proof of proposition 3.5. We will investigate the collection $\{\rho_{ji}\colon U_i \cap U_j \to \mathrm{GA}(k, \mathbb{R}) \mid i, j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$ of transition functions of the affine bundle atlas $\mathcal{B}$ of $\pi\colon E \to M$. For any $i, j \in \mathcal{I}$ and any $p \in U_i \cap U_j$, the map $\rho_{ji}(p)\colon \mathbb{R}^k \to \mathbb{R}^k$ is an affine isomorphism. Hence there exists a unique linear isomorphism $\overrightarrow{\rho_{ji}(p)}\colon \mathbb{R}^k \to \mathbb{R}^k$ such that for any $r \in \mathbb{R}^k$ it holds that $\rho_{ji}(p)(r) = \rho_{ji}(p)(0) + \overrightarrow{\rho_{ji}(p)}(r)$. We thus can define the map

$$\vec{\rho}_{ji}\colon U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R}), \quad p \mapsto \overrightarrow{\rho_{ji}(p)} \tag{3.42}$$

for any $i, j \in \mathcal{I}$ with non-empty bundle chart domain intersection $U_i \cap U_j \neq \emptyset$, providing us with the collection of maps $\{\vec{\rho}_{ji}\colon U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R}) \mid i, j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$.

We would like to promote this collection of maps to bundle chart transition functions of a newly constructed vector bundle. Luckily, we have a theorem at hand whose purpose is precisely that: the Fibre bundle construction theorem. All we have to do is show that $\{\vec{\rho}_{ji}\colon U_i \cap U_j \to \mathrm{GL}(k, \mathbb{R}) \mid i, j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$ meets the hypothesis of the theorem.

Subproof ($\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M). This is immediate from the fact that $\mathcal{B} = \{\alpha_i \mid i \in \mathcal{I}\}$ is an (affine) bundle atlas for $\pi\colon E \to M$.

Subproof (Cocycle conditions). The proof of proposition 3.5 shows that bundle atlas $\mathcal{B}$ is $(\mathrm{GA}(k,\mathbb{R}),)$-compatible. As such, it satisfies the cocycle conditions from proposition 3.1. This means that, one the one hand, for any affine bundle chart $\alpha_i \in \mathcal{B}$ and any point $p \in U_i$ it holds that $\rho_{ii}(p) = \mathrm{id}_{\mathbb{R}^k}$, cf. eq. (3.2). A direct calculation then shows that the associated linear map satisfies $\overrightarrow{\rho_{\alpha\alpha}(p)} = \mathrm{id}_{\mathbb{R}^k}$, which is precisely the identity element of $\mathrm{GL}(k,\mathbb{R})$. On the other hand, for any $i,j,k \in \mathcal{I}$ it holds that $\rho_{ki}(p) = \rho_{kj}(p) \circ \rho_{ji}(p)$, cf. eq. (3.3). Proposition D.1 states that the associated linear map is given by the composition, i.e., $\overrightarrow{\rho_{ki}(p)} = \overrightarrow{\rho_{kj}(p)} \circ \overrightarrow{\rho_{ji}(p)}$. In summary, $\{\vec{\rho}_{ji} \mid i,j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\}$ satisfies the cocycle conditions of the hypothesis of theorem 3.3.

Subproof (Smoothness). It is left to show that for any $i,j \in \mathcal{I}$ that satisfy $U_i \cap U_j \neq \emptyset$ the map $\vec{\rho}_{ij} \colon U_i \cap U_j \to \mathrm{GL}(k,\mathbb{R})$ is smooth. For starters, note that for any $p \in U_i \cap U_j$ and any $1 \leq n \leq k$ it holds that

$$\rho_{ji}(p)(e_n) = \rho_{ji}(p)(0) + \vec{\rho}_{ji}(p)(e_n). \tag{3.43}$$

Given that $\rho_{ji} \colon U_i \cap U_j \to \mathrm{GA}(k,\mathbb{R})$ is smooth, we conclude that for any $1 \leq m,n \leq k$ the map

$$[\vec{\rho}_{ji}]^n{}_m \colon U_i \cap U_j \to \mathbb{R}, \quad p \mapsto (\vec{\rho}_{ji}(p)(e_n))^m = (\rho_{ji}(p)(e_m) - \rho_{ji}(p)(0))^n \tag{3.44}$$

is also smooth. This map, however, is precisely the coordinate-representation of $\vec{\rho}_{ji} \colon U_i \cap U_j \to \mathrm{GL}(k,\mathbb{R})$ with respect to a global chart of $\mathrm{GL}(k,\mathbb{R})$. We conclude that $\vec{\rho}_{ji} \colon U_i \cap U_j \to \mathrm{GL}(k,\mathbb{R})$ is smooth.

This concludes the proof that $\{\vec{\rho}_{ji} \mid i,j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\}$ satisfies the hypotheses of the Fibre bundle construction theorem. In combination with the result of proposition 3.4, this assures that there exists a vector bundle $\vec{\pi} \colon \vec{E} \to M$ with typical fibre $\mathbb{R}^k$ and vector bundle atlas $\vec{\mathcal{B}}$ whose transition functions are given by $\{\vec{\rho}_{ji} \mid i,j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\}$. It remains to show that $\pi \colon E \to M$ is modelled on $\vec{\pi} \colon \vec{E} \to M$.

To this end, we first express the affine structure of the fibres of $\pi \colon E \to M$ in terms of the fibres of $\vec{\pi} \colon \vec{E} \to M$. Let $p \in M$ and $i \in \mathcal{I}$ such that $p \in \pi[\mathrm{Dom}(\alpha_i)]$. Note that there is a vector bundle chart $\vec{\alpha}_i \in \vec{\mathcal{B}}$ associated with the index $i \in \mathcal{I}$ as well. The affine structure of the fibre E_p is the one acquired from the bijection $\alpha|_{E_p} \colon E_p \to \mathbb{R}^k$. We can use the linear isomorphism $\vec{\alpha}_i|_{\vec{E}_p} \colon \vec{E}_p \to \mathbb{R}^k$ in order to treat E_p as an affine space modelled on $\vec{E}_p$. Define

$$\boxplus_p \colon E_p \times \vec{E}_p \to E_p, \quad (a, \vec{a}) \mapsto \alpha_i|_{E_p}^{-1} \left(\alpha_i(a) + \vec{\alpha}_i(\vec{a})\right). \tag{3.45}$$

In fact, the above definition does not depend on the index $i \in \mathcal{I}$ and is thus well-defined. For, if $j \in \mathcal{I}$ is another index such that $p \in U_j$, then

$$\begin{aligned}
\alpha_j|_{E_p}^{-1} \left(\alpha_j(a) + \vec{\alpha}_j(\vec{a})\right) &= \alpha_i|_{E_p}^{-1} \left(\rho_{ij}(p) \left(\alpha_j(a) + \vec{\alpha}_j(\vec{a})\right)\right), && (3.46)\\
&= \alpha_i|_{E_p}^{-1} \left(\rho_{ij}(p) \left(\alpha_j(a)\right) + \vec{\rho}_{ij}(p) \left(\vec{\alpha}_j(\vec{a})\right)\right), && (3.47)\\
&= \alpha_i|_{E_p}^{-1} \left(\alpha_i(a) + \vec{\alpha}_i(\vec{a})\right), && (3.48)
\end{aligned}$$

where we used that $\rho_{ji}(p)$ is affine with $\overrightarrow{\rho_{ji}(p)} = \vec{\rho}_{ji}(p)$. A direct consequence is that for any $a \in E_p$ and any $\vec{a} \in \vec{E}_p$ it holds that

$$\alpha_i \left(a \boxplus_p \vec{a}\right) = \alpha_i(a) + \vec{\alpha}_i(\vec{a}). \tag{3.49}$$

This concludes the proof that $\pi \colon E \to M$ is modelled on $\vec{\pi} \colon \vec{E} \to M$.

Subproof (Uniqueness). Suppose that $\tilde{\pi} \colon \tilde{E} \to M$ is another vector bundle with typical fibre $\tilde{V}$ such that $\pi \colon E \to M$ is modelled on $\tilde{\pi} \colon \tilde{E} \to M$. By lemma 3.6, there exists a bundle atlas $\tilde{\mathcal{B}} = \{\tilde{\alpha} \mid \alpha \in \mathcal{B}\}$ of $\tilde{\pi} \colon \tilde{E} \to M$ such that for any $\alpha \in \mathcal{B}$ it holds that $\tilde{\pi}[\mathrm{Dom}(\tilde{\alpha})] = \pi[\mathrm{Dom}(\alpha)]$ and for any point $p \in \pi[\mathrm{Dom}(\alpha)]$ we have the equality that

$$\forall a \in E_p \colon \forall \tilde{a} \in \tilde{E}_p \colon \quad \alpha|_{E_p}(a \,\tilde{\boxplus}_p\, \tilde{a}) = \alpha|_{E_p}(a) \,\tilde{\boxplus}\, \tilde{\alpha}|_{\tilde{E}_p}(\tilde{a}). \tag{3.50}$$

Let $\phi\colon A \to \mathbb{R}^k$ be an affine isomorphism. It is a diffeomorphism due to item (1) of definition 3.12. Note that there are associated two linear isomorphisms $\vec{\phi}\colon V \to \mathbb{R}^k$ and $\tilde{\phi}\colon \tilde{V} \to \mathbb{R}^k$ that satisfy

$$\forall a \in A\colon \forall v \in V\colon \quad \phi(a \mathbin{\vec{\boxplus}} v) = \phi(a) + \vec{\phi}(v) \tag{3.51}$$

and

$$\forall a \in A\colon \forall \tilde{v} \in \tilde{V}\colon \quad \phi(a \mathbin{\tilde{\boxplus}} \tilde{v}) = \phi(a) + \tilde{\phi}(\tilde{v}), \tag{3.52}$$

respectively. By item (1) of definition 3.15 they are also diffeomorphisms. Consequently, there exists a canonical linear diffeomorphism between the typical fibres V and $\tilde{V}$

$$I := \tilde{\phi}^{-1} \circ \vec{\phi}\colon V \to \tilde{V}. \tag{3.53}$$

It is canonical in the sense that it does not depend on the choice of the affine isomorphism $\phi\colon A \to \mathbb{R}^k$. We can now define the map

$$\Phi\colon \vec{E} \to \tilde{E}, \quad \vec{a} \mapsto (\tilde{\pi}, \tilde{\alpha})^{-1} \circ (\mathrm{id}_M, I) \circ (\vec{\pi}, \vec{\alpha})(\vec{a}) \quad \text{for } \alpha \in \mathcal{B}\colon \vec{a} \in \vec{\pi}[\mathrm{Dom}(\vec{\alpha})], \tag{3.54}$$

which is well-defined since it can easily be checked that $\Phi(\vec{a}) = (a \mathbin{\vec{\boxplus}}_p \vec{a}) \mathbin{\tilde{\boxminus}}_p a$ holds for all $a \in E_p$ and all $\vec{a} \in \vec{E}_p$, where $p \in M$. Since $I\colon V \to \tilde{V}$ is a diffeomorphism, the map $\Phi\colon \vec{E} \to \tilde{E}$ turns out to be a diffeomorphism. By definition, the map Φ satisfies $\tilde{\pi} \circ \Phi = \vec{\pi}$ and therefore qualifies as a bundle isomorphism along id_M. It now suffices to read off that the restriction

$$\Phi|_{\vec{E}_p} = \tilde{\alpha}|_{\tilde{E}_p}^{-1} \circ I \circ \vec{\alpha}|_{\vec{E}_p}\colon \vec{E}_p \to \tilde{E}_p \tag{3.55}$$

is a linear isomorphism, in order to conclude the proof that $\Phi\colon \vec{E} \to \tilde{E}$ is a vector bundle isomorphism along id_M.

Remark 3.7. For the fans of category theory: The first part of the above theorem can be used in order to define a functor from the category of affine bundles to the category of vector bundles.

Lemma 3.8 (Subtraction and addition of local sections). *Let $\pi\colon E \to M$ be an affine bundle modelled on $\vec{\pi}\colon \vec{E} \to M$. The subtraction*

$$s_1 \underset{\Gamma(E|_U)}{\boxminus} s_2\colon U \to E, \quad p \mapsto s_1(p) \underset{p}{\boxminus} s_2(p) \tag{3.56}$$

of two local sections $s_1\colon U \to E$ and $s_2\colon U \to E$ over $U \in \mathcal{O}_M$ is a local section of $\vec{\pi}\colon \vec{E} \to M$. Similarly, the addition

$$s \underset{\Gamma(E|_U)}{\boxplus} \vec{s}\colon U \to E, \quad p \mapsto s(p) \underset{p}{\boxplus} \vec{s}(p) \tag{3.57}$$

of local sections $s\colon U \to E$ and $\vec{s}\colon U \to E$ over $U \in \mathcal{O}_M$ of $\pi\colon E \to M$ and $\vec{\pi}\colon \vec{E} \to M$, respectively, is a local section of $\pi\colon E \to M$.

Proof 3.8. Without loss of generality, assume that the typical fibres are chosen to be $\mathbb{R}^k$ for suitable $k \in \mathbb{N}$. Let $\mathcal{B}$ be an affine bundle atlas for $\pi\colon E \to M$ and let $\vec{\mathcal{B}} = \{\vec{\alpha} \mid \alpha \in \mathcal{B}\}$ be the derived vector bundle atlas for $\vec{\pi}\colon \vec{E} \to M$ such that for any $\alpha \in \mathcal{B}$ it holds that $\vec{\pi}[\mathrm{Dom}(\vec{\alpha})] = \pi[\mathrm{Dom}(\alpha)]$ and for any point $p \in \pi[\mathrm{Dom}(\alpha)]$ we have the equality $\vec{\alpha}|_{\vec{E}_p} = \overrightarrow{\alpha|_{E_p}}$.

Subproof (Subtraction). Let $\alpha \in \mathcal{B}$. For $p \in \pi[\mathrm{Dom}(\alpha)]$ we have

$$\vec{\alpha} \circ \left(s_1 \underset{\Gamma(E|_U)}{\boxminus} s_2 \right)(p) = \vec{\alpha}\left(s_1(p) \underset{p}{\boxminus} s_2(p) \right) \tag{3.58}$$

$$= \alpha \circ s_1(p) - \alpha \circ s_2(p), \tag{3.59}$$

which clearly depends smoothly on p. This proves the first part.

Subproof (Addition). Let $\alpha \in \mathcal{B}$. For $p \in \pi[\mathrm{Dom}(\alpha)]$ we have

$$\alpha \circ \left(s \underset{\Gamma(E|_U)}{\boxplus} \vec{s} \right)(p) = \alpha \left(s(p) \underset{p}{\boxplus} \vec{s}(p) \right) \tag{3.60}$$

$$= \alpha \circ s(p) + \vec{\alpha} \circ \vec{s}(p), \tag{3.61}$$

which clearly depends smoothly on p. This concludes the second part.

Corollary 3.9. *An immediate consequence is that the affine combination*

$$\sum_{i=1..k}^{\boxplus_{\Gamma(E|_U)}} \lambda_i s_i \colon U \to E, \quad p \mapsto \sum_{i=1..k}^{\boxplus_p} \lambda_i(p) s_i(p) \tag{3.62}$$

of local sections $s_1, \dots, s_k \colon U \to E$, with respect to the weights $\lambda_1, \dots, \lambda_k \in C^\infty(U)$ that satisfy

$$\forall p \in U \colon \sum_{i=1..k} \lambda_i(p) = 1, \tag{3.63}$$

is a local section of $\pi \colon E \to M$. Refer to Definition D.4 (Affine combination).

Proposition 3.10. *Every affine bundle admits a global section.*

Proof 3.10. Let $\pi \colon E \to M$ be an affine bundle of rank k. For simplicity, let us assume that the typical fibre $\mathbb{R}^k$ was chosen, cf. proof of proposition 3.5. Let $\mathcal{B}$ be an affine bundle atlas of $\pi \colon E \to M$. Since M is paracompact, we can assume without loss of generality that the open cover $\mathcal{U} := \{\pi[\mathrm{Dom}(\alpha)] \mid \alpha \in \mathcal{B}\}$ is locally finite. Furthermore, there exists a partition of unity $\{\lambda_\alpha \mid \alpha \in \mathcal{B}\}$ subordinate to $\mathcal{U}$. For each $\alpha \in \mathcal{B}$, the map

$$X_\alpha \colon \pi[\mathrm{Dom}(\alpha)] \to E, \quad p \mapsto (\pi, \alpha)^{-1}(p, 0) \tag{3.64}$$

is a local section. Using the partition of unity $\{\lambda_\alpha \mid \alpha \in \mathcal{B}\}$ we can define the map

$$X \colon M \to E, \quad p \mapsto \sum_{\alpha \in \mathcal{B} \colon \lambda_\alpha(p) \neq 0}^{\boxplus_{E_p}} \lambda_\alpha(p) X_\alpha(p), \tag{3.65}$$

where we used the affine combination in each fibre E_p. Fix a point $p \in M$. We show that $X \colon M \to E$ is smooth at p. Since $\mathcal{U}$ is locally finite, there exists a neighbourhood $W_p \in \mathcal{O}_M$ of p that intersects only finitely many elements of $\mathcal{U}$. Define

$$\tilde{W}_p := \bigcap \{\pi[\mathrm{Dom}(\alpha)] \cap W_p \mid \alpha \in \mathcal{B} \colon p \in \pi[\mathrm{Dom}(\alpha)]\}. \tag{3.66}$$

This a neighbourhood of p due to the fact that it is a finite intersection of neighbourhoods of p. The restriction of X to $\tilde{W}_p$ is given by

$$X|_{\tilde{W}_p} = \sum_{\alpha \in \mathcal{B} \colon \lambda_\alpha|_{\tilde{W}_p} \neq 0}^{\boxplus_{\Gamma(E|_{\tilde{W}_p})}} \lambda_\alpha X_\alpha. \tag{3.67}$$

This, however, is a smooth map due to corollary 3.9. We have constructed a global section of $\pi \colon E \to M$.

Proposition 3.11. *Let $\pi \colon E \to M$ be an affine bundle modelled on a vector bundle $\vec{\pi} \colon \vec{E} \to M$. Then there exists an affine bundle isomorphism between $\vec{\pi} \colon \vec{E} \to M$ and $\pi \colon E \to M$.*

Proof 3.11. Let $s\colon M \to E$ be a global section of $\pi\colon E \to M$. Its existence is guaranteed by proposition 3.10. Given the global section $s\colon M \to E$, it is straightforward to propose a derived candidate for the desired affine bundle isomorphism

$$\Phi_s\colon \vec{E} \to E, \quad \vec{a} \mapsto s\left(\vec{\pi}(\vec{a})\right) \underset{\vec{\pi}(\vec{a})}{\boxplus} \vec{a}. \tag{3.68}$$

It is straightforward to check that it is indeed an affine bundle isomorphism. By definition, we implemented that $\pi \circ \Phi_s = \vec{\pi}$. In order to verify that Φ_s is smooth at any point $\vec{a} \in \vec{E}$, pick an affine bundle chart α of $\pi\colon E \to M$ such that $\vec{\pi}(\vec{a}) \in \pi[\mathrm{Dom}(\alpha)]$. Denote by $\vec{\alpha}$ the vector bundle chart of $\vec{\pi}\colon \vec{E} \to M$ that comes associated to α by means of lemma 3.6. Then for any $p \in \pi[\mathrm{Dom}(\alpha)]$ and any $\xi \in v$ we have

$$\begin{aligned}
\alpha \circ \Phi_s \circ (\vec{\pi}, \vec{\alpha})^{-1}(p, \xi) &= \alpha|_{E_p}\left(s \circ \vec{\pi} \circ (\vec{\pi}, \vec{\alpha})^{-1}(p, \xi) \boxplus_p (\vec{\pi}, \vec{\alpha})^{-1}(p, \xi)\right) &(3.69)\\
&= \alpha \circ s(p) \boxplus \overrightarrow{\alpha|_{E_p}}\left((\vec{\pi}, \vec{\alpha})^{-1}(p, \xi)\right) &(3.70)\\
&= \alpha \circ s(p) \boxplus \xi, &(3.71)
\end{aligned}$$

where we used that $\overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}$. Since $\boxplus$ is smooth, by definition 3.15, it follows that Φ_s is indeed smooth. The above equality lets us directly verify that Φ_s is fibre-wisely affine. For if $\xi, \eta \in V$, then

$$\begin{aligned}
\alpha \circ \Phi_s \circ (\vec{\pi}, \vec{\alpha})^{-1}(p, \xi + \eta) &= \alpha \circ s(p) \boxplus (\xi + \eta) &(3.72)\\
&= (\alpha \circ s(p) \boxplus \xi) \boxplus \eta &(3.73)\\
&= \alpha \circ \Phi_s \circ (\vec{\pi}, \vec{\alpha})^{-1}(p, \xi) \boxplus \eta. &(3.74)
\end{aligned}$$

This concludes the proof that $\Phi_s\colon \vec{E} \to E$ is an affine bundle morphism. It is left to check that it has a smooth inverse. Its inverse is readily found to be

$$\Phi_s^{-1}\colon E \to \vec{E}, \quad a \mapsto a \underset{\pi(a)}{\boxminus} s(\pi(a)). \tag{3.75}$$

It is smooth at $a \in \mathrm{Dom}(\alpha)$ since for any $p \in \pi[\mathrm{Dom}(\alpha)]$ and any $\xi \in A$ we have that

$$\begin{aligned}
\vec{\alpha} \circ \Phi_s^{-1} \circ (\pi, \alpha)^{-1}(p, \xi) &= \vec{\alpha}\left((\pi, \alpha)^{-1}(p, \xi) \underset{p}{\boxminus} s \circ \pi \circ (\pi, \alpha)^{-1}(p, \xi)\right) &(3.76)\\
&= \overrightarrow{\alpha|_{E_p}}\left((\pi, \alpha)^{-1}(p, \xi) \underset{p}{\boxminus} s(p)\right) &(3.77)\\
&= \alpha|_{E_p}\left((\pi, \alpha)^{-1}(p, \xi)\right) \boxminus \alpha|_{E_p}(s(p)) &(3.78)\\
&= \xi \boxminus \alpha \circ s(p) &(3.79)
\end{aligned}$$

depends smoothly on $\xi \in A$ and $p \in M$. Note that we used $\overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}$. This concludes the proof.

Corollary 3.12. *An affine bundle modelled on a trivial vector bundle is trivial.*

Remark 3.8. The fact that any affine bundle admits a global section allows us to understand affine bundles entirely in terms of the vector bundles that they are modelled on.

Proposition 3.11 tells us that every affine bundle $\pi\colon E \to M$ can be made into a vector bundle by prescription of a global section as null section. Viewing affine bundles as $(\mathrm{GA}(k, \mathbb{R}),)$-bundles, the result tells us that it is always possible to reduce the structure group from $\mathrm{GA}(k, \mathbb{R})$ to its Lie subgroup $\mathrm{GL}(k, \mathbb{R})$.

Observe that the resulting vector bundle $\vec{\pi}\colon \vec{E} \to M$ is such that $\pi\colon E \to M$ is modelled on $\vec{\pi}\colon \vec{E} \to M$.

4 Principal bundles

Definition 4.1. Let $\triangleleft\colon M \times G \to M$ and $\blacktriangleleft\colon M' \times G' \to M'$ be two Lie group right actions and let $\rho\colon G \to G'$ be a Lie group homomorphism.

A smooth map $f\colon M \to M'$ is said to be ρ**-equivariant** if the diagram

$$\begin{tikzcd} M \times G \arrow[r, "f\times\rho"] \arrow[d, "\triangleleft"'] & M' \times G' \arrow[d, "\blacktriangleleft"] \\ M \arrow[r, "f"'] & M' \end{tikzcd}$$

commutes. That is, if

$$\forall g \in G\colon \forall p \in M\colon f(p \triangleleft g) = f(p) \blacktriangleleft \rho(g). \tag{4.1}$$

Definition 4.2 (Principal bundle)**.** Let $\pi\colon P \to M$ be a fibre bundle with typical fibre given by a Lie group $(G, \mathcal{O}_G, \mathcal{A}_G, \bullet)$ and let $\triangleleft\colon P \times G \to P$ be a Lie group right action. $\pi\colon P \to M$ is said to be a **principal bundle with respect to** $\triangleleft$ if:

1. the Lie group right action $\triangleleft$ is free,
2. the Lie group right action $\triangleleft$ is fibre-preserving, i.e.,
$$\forall g \in G\colon \quad \pi \circ (\triangleleft g) = \pi, \tag{4.2}$$
3. there exists a bundle atlas $\mathcal{B}$ for $\pi\colon P \to M$, called **principal bundle atlas**, with the property that each (**principal**) bundle chart $\alpha \in \mathcal{B}$ is id_G-equivariant, i.e.,
$$\forall a \in \mathrm{Dom}(\alpha)\colon \forall g \in G\colon \quad \alpha(a \triangleleft g) = \alpha(a) \bullet g. \tag{4.3}$$

Lemma 4.1. *A principal bundle atlas $\mathcal{B}$ is $(G, \bullet)$-compatible, with $\bullet\colon G \times G \to G$ regarded as a left action.*

Proof 4.1. First note that the Lie group left action $\bullet\colon G \times G \to G$ is effective, since $(G, \bullet)$ is a group.

Let $\alpha, \beta \in \mathcal{B}$ such that $\pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)] \neq \emptyset$. Then define the map

$$\rho_{\beta\alpha}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\alpha)] \to G, \quad p \mapsto \beta \circ (\pi, \alpha)^{-1}(p, e). \tag{4.4}$$

Note that this map is smooth by definition. Let $p \in \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)]$ and $g \in G$. Observe that

$$\begin{aligned} (\pi, \alpha)\left((\pi, \alpha)^{-1}(p, e) \triangleleft g\right) &= \left(p, \alpha\left((\pi, \alpha)^{-1}(p, e)\right) \bullet g\right) && (4.5) \\ &= (p, g), && (4.6) \end{aligned}$$

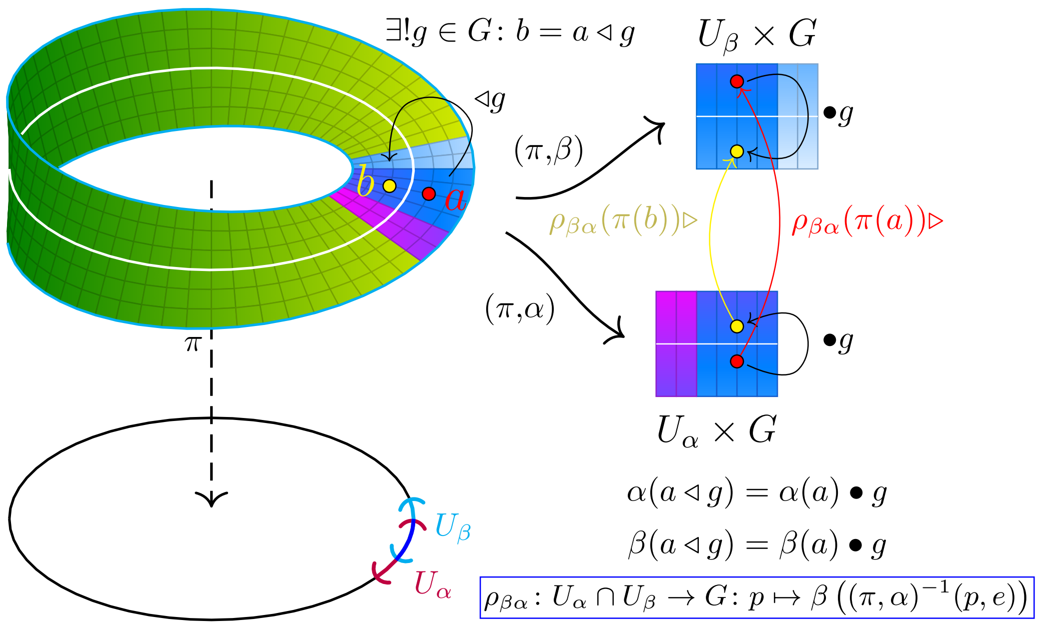

Figure 4.1: If we remove a section from the Möbius strip, we obtain a principal bundle with structure group $\mathrm{GL}(1,\mathbb{R})$. In fact, it is the principal bundle associated to the Möbius strip interpreted as a vector bundle from fig. 3.2.

where we used items (2) and (3) from definition 4.2. Consequently,

$$(\pi,\alpha)^{-1}(p,g) = (\pi,\alpha)^{-1}(p,e) \triangleleft g. \tag{4.7}$$

We can use this result in order to deduce that

$$\begin{aligned} \beta \circ (\pi,\alpha)^{-1}(p,g) &= \beta\left((\pi,\alpha)^{-1}(p,e) \triangleleft g\right) && (4.8)\\ &= \beta\left((\pi,\alpha)^{-1}(p,e)\right) \bullet g && (4.9)\\ &= \rho_{\beta\alpha}(p) \bullet g, && (4.10) \end{aligned}$$

where we used item (3) once more and recognized our previously defined map $\rho_{\beta\alpha}$. This concludes the proof.

Corollary 4.2. *A principal bundle $\pi\colon P \to M$ with respect to a Lie group right action $\triangleleft\colon P \times G \to P$ is a fibre bundle with effective structure group action with respect to left translation $\bullet\colon G \times G \to G$.*

Theorem 4.3 (Principal bundle of the translation group is trivial)**.** *Suppose that $\pi\colon E \to M$ is a principal bundle with respect to a Lie group right action $\triangleleft\colon E \times \mathbb{R}^k \to E$ of the translation group $(\mathbb{R}^k,+)$.*

$\pi\colon E \to M$ is an affine bundle modelled on a trivial vector bundle.

Consequently, by corollary 3.12, $\pi\colon E \to M$ is trivial.

Proof 4.3. Let $\mathcal{B}$ a principal bundle atlas for $\pi\colon E \to M$. By lemma 4.1, $\mathcal{B}$ is $(\mathbb{R}^k,+)$-compatible. Since the general affine group $\mathrm{GA}(k,\mathbb{R}) = \mathbb{R}^k \rtimes \mathrm{GL}(k,\mathbb{R})$ can be expressed as the semidirect product of $\mathbb{R}^k$ by the general linear group $\mathrm{GL}(k,\mathbb{R})$, it is also true that $\mathcal{B}$ is $(\mathrm{GA}(k,\mathbb{R}),)$-compatible. We established that $\pi\colon E \to M$ is an affine bundle. Theorem 3.7 states that there exists a vector bundle $\vec{\pi}\colon \vec{E} \to M$ such that $\pi\colon E \to M$ is modelled on $\vec{\pi}\colon \vec{E} \to M$. Moreover, lemma 3.6 guarantees the existence of a vector bundle atlas $\vec{\mathcal{B}} = \{\vec{\alpha} \mid \alpha \in \mathcal{B}\}$ for $\vec{\pi}\colon \vec{E} \to M$ with the property that for every $\alpha \in \mathcal{B}$ it holds that $\pi[\mathrm{Dom}(\alpha)] = \vec{\pi}[\mathrm{Dom}(\vec{\alpha})]$ and such that for every point $p \in \pi[\mathrm{Dom}(\alpha)]$ we have the equality $\overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}$.

Let us take a closer look at the induced vector bundle atlas $\vec{\mathcal{B}}$. We claim that we can build a global vector bundle chart for $\vec{\pi}\colon \vec{E} \to M$ starting from the vector bundle atlas $\vec{\mathcal{B}}$. For, if $\alpha, \beta \in \mathcal{B}$ are principal bundle charts of $\pi\colon E \to M$ then for any point $p \in \pi[\mathrm{Dom}(\alpha)] \cap \pi[\mathrm{Dom}(\beta)]$ and any $v \in \mathbb{R}^k$ it holds that

$$\left(\beta|_{E_p} \circ \alpha|_{E_p}^{-1}\right)(v) = \left(\beta|_{E_p} \circ \alpha|_{E_p}\right)(0) + v, \tag{4.11}$$

where we used that $\mathcal{B}$ is $(\mathbb{R}^k, +)$-compatible. By direct comparison, we read off that

$$\vec{\beta}|_{\vec{E}_p} \circ \vec{\alpha}|_{\vec{E}_p}^{-1} = \overrightarrow{\beta|_{E_p}} \circ \overrightarrow{\alpha|_{E_p}}^{-1} = \overrightarrow{\beta|_{E_p} \circ \alpha|_{E_p}^{-1}} = \mathrm{id}_{\mathbb{R}^k}, \tag{4.12}$$

where we used that $\overrightarrow{\alpha|_{E_p}} = \vec{\alpha}|_{\vec{E}_p}$ and $\overrightarrow{\beta|_{E_p}} = \vec{\beta}|_{\vec{E}_p}$. This tells us that all the vector bundle charts of $\vec{\mathcal{B}}$ agree on their respective domain overlaps. Thus we can extend the vector bundle charts of $\vec{\mathcal{B}}$ to a global vector bundle chart or, alternatively, define the vector bundle isomorphism

$$\Phi\colon M \times \mathbb{R}^k \to \vec{E}, \quad (p, v) \mapsto (\vec{\pi}, \vec{\alpha})^{-1}(p, v). \tag{4.13}$$

This proves that $\vec{\pi}\colon \vec{E} \to M$ is trivial. Consequently, $\pi\colon E \to M$ is also trivial.

Remark 4.1. Theorem 4.3 is the first central result of this work. Applied to teleparallel gravity in particular it states that the principal bundle P of teleparallel gravity (with structure group given by the translation group $\mathbb{R}^4$) is trivial. Given a affine bundle isomorphism (in this context usually referred to as solder form) between this principal bundle P and the tangent bundle TM of the base manifold M, it follows that the tangent bundle TM itself is trivial. In other words, the base manifold M is parallelizable.

The rest of this work will discuss whether or not the result of theorem 4.3 represents some obstruction to the applicability of teleparallel gravity. To this end, we will continue introducing more concepts in order to be able to define covariant derivative operators and parallel transport systems on vector bundles. This will allow us to discuss the holonomy of the tangent bundle of the base manifold M. We will see that for a simply-connected spacetime M, a curvature-free metric-compatible covariant derivative operator exists if and only if a global orthonormal frame exists. Note that the latter is a stronger condition than parallelizability.

Finally we discuss that a non-compact spacetime admits a spin structure if and only if a global orthonormal frame exists. Even in the non-simply-connected case, this means that a teleparallel formulation is possible for all spacetimes that admit a spin structure.

For now, we will resume our journey with the construction of the most prominent principal bundle that we will encounter. The frame bundle of a vector bundle.

Proposition 4.4. *Let $\pi\colon E \to M$ be a vector bundle of rank k with vector bundle atlas $\mathcal{B}$. For every point $p \in M$, denote by $\mathrm{Fr}(E)_p$ the set of bases of E_p.*

Define the set

$$\mathrm{Fr}(E) := \bigcup \{\mathrm{Fr}(E)_p \mid p \in M\}, \tag{4.14}$$

a projection on $\mathrm{Fr}(E)$ given by

$$\hat{\pi}\colon \mathrm{Fr}(E) \to M, \quad (e_1, \ldots, e_k) \mapsto p \text{ such that } (e_1, \ldots, e_k) \text{ is a basis of } E_p, \tag{4.15}$$

a right action of $\mathrm{GL}(k, \mathbb{R})$ on $\mathrm{Fr}(E)$ given by

$$\triangleleft\colon \mathrm{Fr}(E) \times \mathrm{GL}(k, \mathbb{R}) \to \mathrm{Fr}(E), \quad ((e_1, \ldots, e_k), A) \mapsto (A^m{}_1 e_m, \ldots, A^m{}_k e_m), \tag{4.16}$$

and a principal bundle atlas $\hat{\mathcal{B}}$ given by

$$\hat{\mathcal{B}} := \left\{\alpha \times \cdots \times \alpha\colon \hat{\pi}^{-1}[\pi[\mathrm{Dom}(\alpha)]] \mapsto \mathrm{GL}(k, \mathbb{R}) \mid \alpha \in \mathcal{B}\right\}. \tag{4.17}$$

Then $\hat{\pi}\colon \operatorname{Fr}(E) \to M$ is a principal bundle with respect to $\triangleleft\colon \operatorname{Fr}(E) \times \mathrm{GL}(k,\mathbb{R}) \to \operatorname{Fr}(E)$.

Proof 4.4. This is example is important enough for us to go through the details of theorem 2.1.

Subproof ($\hat{\pi}\colon \operatorname{Fr}(E) \to M$ is a surjection). Let $p \in M$. Since $\pi\colon E \to M$ is a surjection, the fibre E_p over p is non-empty. Every vector space admits a Hamel basis. (A zero-dimensional vector space has the empty set as basis.) Hence there exists an element $(e_1, \dots, e_k) \in \operatorname{Fr}(E)_p$ such that $\hat{\pi}\left((e_1, \dots, e_k)\right) = p$.

Subproof ($\hat{\mathcal{B}}$ is a proto bundle atlas and $(G, \bullet)$-compatible). We have to check items 1 to 4 from theorem 2.1. Since $\hat{\pi}\colon \operatorname{Fr}(E) \to M$ is a surjection, for any $\alpha \in \mathcal{B}$ it holds that

$$\hat{\pi}\left[\hat{\pi}^{-1}\left[\pi[\operatorname{Dom}(\alpha)]\right]\right] = \pi[\operatorname{Dom}(\alpha)]. \tag{4.18}$$

This is enough for us to verify items 1 and 2. That are, for any $\hat{\alpha} \in \hat{\mathcal{B}}$ it holds that $\operatorname{Dom}(\hat{\alpha}) = \hat{\pi}^{-1}[\hat{\pi}[\operatorname{Dom}(\hat{\alpha})]]$ and the collection $\{\hat{\pi}[\operatorname{Dom}(\hat{\alpha})] \mid \hat{\alpha} \in \hat{\mathcal{B}}\}$ is an open cover of M.

Next let $\hat{\alpha} \in \mathcal{B}$ and $\alpha \in \mathcal{B}$ such that $\hat{\alpha} = \alpha \times \dots \times \alpha$. Let us convince ourselves that the map

$$\begin{aligned} (\hat{\pi}, \hat{\alpha})\colon \operatorname{Dom}(\hat{\alpha}) &\to \hat{\pi}[\operatorname{Dom}(\hat{\alpha})] \times \mathrm{GL}(k,\mathbb{R}), \\ (e_1, \dots, e_k) &\mapsto \left(\pi(e_1), \begin{pmatrix} \alpha^1(e_1) & \dots & \alpha^1(e_k) \\ \vdots & \ddots & \vdots \\ \alpha^k(e_1) & \dots & \alpha^k(e_k) \end{pmatrix}\right) \end{aligned} \tag{4.19}$$

is a bijection. To this end, it suffices to remember that for any point $p \in M$ the map $\alpha|_{E_p}\colon E_p \to \mathbb{R}^k$ is a linear isomorphism and note that

$$\begin{aligned} \Phi_\alpha\colon \hat{\pi}[\operatorname{Dom}(\hat{\alpha})] \times \mathrm{GL}(k,\mathbb{R}) &\to \operatorname{Dom}(\hat{\alpha}), \\ (p, A) &\mapsto \left(\alpha|_{E_p}^{-1}\left(A^1{}_1, \dots, A^k{}_1\right), \dots, \alpha|_{E_p}^{-1}\left(A^1{}_k, \dots, A^k{}_k\right)\right) \end{aligned} \tag{4.20}$$

is the desired inverse of $(\hat{\pi}, \hat{\alpha})$. Φ_α is well-defined due to the fact that for any $A \in \mathrm{GL}(k,\mathbb{R})$ the columns of A form a basis for $\mathbb{R}^k$ and due to $\alpha|_{E_p}\colon E_p \to \mathbb{R}^k$ being a linear isomorphism. Because then $\Phi_\alpha(p, A)$ is a basis of E_p.

Finally, let us tackle item 4. Let $\hat{\alpha}, \hat{\beta} \in \hat{\mathcal{B}}$ and $\alpha, \beta \in \mathcal{B}$ such that $\hat{\alpha} = \alpha \times \dots \times \alpha$ and $\hat{\beta} = \beta \times \dots \times \beta$. We want to show that

$$\begin{aligned} \left(\hat{\pi}, \hat{\beta}\right) \circ (\hat{\pi}, \hat{\alpha})^{-1} : \hat{\pi}[\operatorname{Dom}(\hat{\alpha}) \cap \operatorname{Dom}(\hat{\beta})] \times \mathrm{GL}(k,\mathbb{R}) &\to \hat{\pi}[\operatorname{Dom}(\hat{\alpha}) \cap \operatorname{Dom}(\hat{\beta})] \times \mathrm{GL}(k,\mathbb{R}) \\ (p, A) &\mapsto \left(p, \left(\beta|_{E_p} \circ \alpha|_{E_p}^{-1}\left(A^1{}_k, \dots, A^k{}_1\right), \dots, \beta|_{E_p} \circ \alpha|_{E_p}^{-1}\left(A^k{}_1, \dots, A^k{}_k\right)\right)\right) \end{aligned} \tag{4.21}$$

is a smooth. We immediately recognize the $\mathrm{GL}(k,\mathbb{R})$-valued transition function $\rho_{\beta\alpha}$ of the vector bundle atlas $\mathcal{B}$. For $1 \leq i \leq k$. we have

$$\beta|_{E_p} \circ \alpha|_{E_p}^{-1}\left(A^1{}_i, \dots, A^k{}_i\right) = \left(\rho_{\beta\alpha}(p)^1{}_m A^m{}_i, \dots, \rho_{\beta\alpha}(p)^k{}_m A^m{}_i\right). \tag{4.22}$$

Here we recognize the group operation

$$\begin{aligned} \bullet\colon \mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(k,\mathbb{R}) &\to \mathrm{GL}(k,\mathbb{R}), \\ (B, C) &\mapsto \left(B^1{}_m C^m{}_1, \dots, B^k{}_m C^m{}_1, \dots, B^1{}_m C^m{}_k, \dots, B^k{}_m C^m{}_k\right) \end{aligned} \tag{4.23}$$

Altogether, we obtain

$$\left(\hat{\pi}, \hat{\beta}\right) \circ (\hat{\pi}, \hat{\alpha})^{-1} : (p, A) \mapsto (p, \rho_{\beta\alpha}(p) \bullet A), \tag{4.24}$$

which is smooth since $\rho_{\beta\alpha}$ and $\bullet$ are smooth. This also already proves that $\hat{\mathcal{B}}$ is $(G, \bullet)$-compatible.

Subproof ($\hat{\alpha} \in \hat{\mathcal{B}}$ is id_G-equivariant). Let $\hat{\alpha} \in \mathcal{B}$ and $\alpha \in \mathcal{B}$ such that $\hat{\alpha} = \alpha \times \cdots \times \alpha$. Then for any $A \in \mathrm{GL}(k, \mathbb{R})$ and $(e_1, \ldots, e_k) \in \mathrm{Dom}(\hat{\alpha})$:

$$\begin{aligned}\hat{\alpha}\left((e_1, \ldots, e_k) \triangleleft A\right) &= \hat{\alpha}\left((A^m{}_1 e_m, \ldots, A^m{}_k e_m)\right) \\ &= (\alpha(A^m{}_1 e_m), \ldots, \alpha(A^m{}_k e_m)) \\ &= \begin{pmatrix} A^m{}_1 \alpha^1(e_m) & \ldots & A^m{}_k \alpha^1(e_m) \\ \vdots & \ddots & \vdots \\ A^m{}_1 \alpha^k(e_m) & \ldots & A^m{}_k \alpha^k(e_m) \end{pmatrix} \\ &= \hat{\alpha}\left(e_1, \ldots, e_k\right) \bullet A.\end{aligned}$$

This proves that $\hat{\alpha}$ is id_G-equivariant. It is left to show that $\triangleleft$ is indeed a Lie group right action.

Subproof ($\triangleleft$ is a free and fibre-preserving Lie group right action). Let us first show that $\triangleleft$ is smooth. We can make use of the result of the last item. Simply set $(e_1, \ldots, e_k) = (\hat{\pi}, \hat{\alpha})^{-1}(p, B)$ for some $p \in \hat{\pi}[\mathrm{Dom}(\hat{\alpha})]$ and $B \in \mathrm{GL}(k, \mathbb{R})$. The last equation then becomes

$$(\hat{\pi}, \hat{\alpha}) \left((\hat{\pi}, \hat{\alpha})^{-1} (p, B) \triangleleft A \right) = (p, B \bullet A). \tag{4.25}$$

This proves that $\triangleleft$ is smooth.

It is fairly straightforward to check that $\triangleleft\colon \mathrm{Fr}(E) \times \mathrm{GL}(k, \mathbb{R}) \to \mathrm{Fr}(E)$ is a right action. By design, $\triangleleft$ also preserves fibres. That is, for every $g \in G$, it holds that $\hat{\pi} \circ (\triangleleft g) = \hat{\pi}$. It is left to show that $\triangleleft$ is free.

Let $(e_1, \ldots, e_k) \in \mathrm{Fr}(E)$ and $A \in \mathrm{GL}(k, \mathbb{R})$ such that $(e_1, \ldots, e_k) \triangleleft A = (e_1, \ldots, e_k)$. That is, for $1 \leq i \leq k$ it holds that

$$A^m{}_i e_m = e_i. \tag{4.26}$$

But $(e_1, \ldots, e_k)$ is a basis of $E_{\hat{\pi}((e_1, \ldots, e_k))}$. Thus A is the identity of $\mathrm{GL}(k, \mathbb{R})$ and, consequently, $\triangleleft$ is free.

This concludes the proof that $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ is a principal bundle with respect to the Lie group right action $\triangleleft\colon \mathrm{Fr}(E) \times \mathrm{GL}(k, \mathbb{R}) \to \mathrm{Fr}(E)$.

Definition 4.3 (Frame bundle)**.** $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ is called the **frame bundle** of $\pi\colon E \to M$.

Example 4.1. Observe that the Möbius strip as a principal bundle from fig. 4.1 is the frame bundle of the Möbius strip as a vector bundle from fig. 3.2.

Remark 4.2. As seen above, constructing a fibre bundle, let alone a principal bundle, is a fair amount of work.

In a very similar fashion in which we established the Fibre bundle construction theorem in order to facilitate the construction of fibre bundles with effective structure group action, we will now pave the way towards the Principal bundle construction theorem below.

Definition 4.4 (Structure group reduction)**.** Let $\pi\colon P \to M$ be a principal bundle with respect to the Lie group right action $\triangleleft\colon P \times G \to P$. Let $P' \subseteq P$ be an embedded submanifold of P and let $G' \subseteq G$ be a Lie subgroup of G.

The restriction $\pi|_{P'}\colon P' \to M$ is said to be a **structure group reduction from G to G'** of $\pi\colon P \to M$ if it is a principal bundle with respect to the restriction $\triangleright|_{G' \times P'}\colon P' \times G' \to P$.

We shall encounter some structure group reductions in chapter 7.

Definition 4.5 (Principal bundle morphism)**.** Let $\pi\colon P \to M$ and $\pi'\colon P' \to M'$ be two principal bundles with respect to the Lie group right actions $\triangleleft\colon P \times G \to P$ and $\blacktriangleleft\colon P' \times G' \to P'$, respectively. Suppose we are given two smooth maps $\Phi\colon P \to P'$ and $\varphi\colon M \to M'$ and a Lie group homomorphism $\rho\colon G \to G'$.

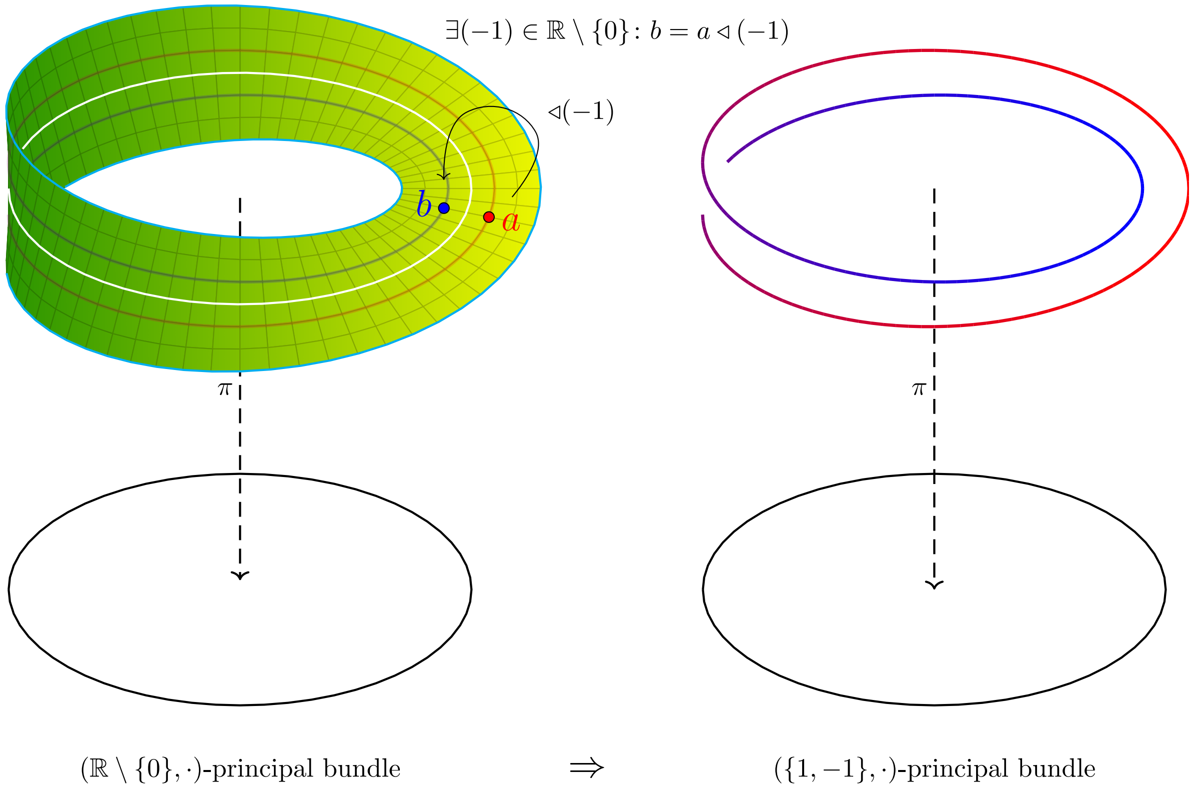

Figure 4.2: An exemplary visualization of a structure group reduction using the example of a Möbius strip as principal bundle from fig. 4.1. The fact that the maximal structure group reduction has structure group $(\{-1, 1\}, \cdot)$ reflects the observation that the Möbius strip as a vector bundle (cf. fig. 3.2) is not orientable.

We say that Φ is a **ρ-principal bundle morphism along φ from $\pi \colon P \to M$ to $\pi' \colon P' \to M'$** if it is a bundle morphism along φ and ρ-equivariant, i.e.,

$$\forall a \in P \colon \forall g \in G \colon \quad \Phi(a \triangleleft g) = \Phi(a) \blacktriangleleft \rho(g). \tag{4.27}$$

Definition 4.6 (Principal bundle isomorphism)**.** Let $\pi \colon P \to M$ and $\pi' \colon P' \to M'$ be two principal bundles with respect to Lie group right actions $\triangleleft \colon P \times G \to P$ and $\blacktriangleleft \colon P' \times G' \to P'$, respectively. Suppose we are given two diffeomorphisms $\Phi \colon P \to P'$ and $\varphi \colon M \to M'$ and a Lie group isomorphism $\rho \colon G \to G'$.

We say that Φ is a **ρ-principal bundle isomorphism along φ between $\pi \colon P \to M$ and $\pi' \colon P' \to M'$** if Φ is a ρ-principal bundle morphism along φ and Φ^{-1} is a ρ^{-1}-principal bundle morphism along φ^{-1}.

Lemma 4.5. *Let $\pi \colon P \to M$ and $\pi' \colon P' \to M$ be two principal bundles over M with respect to Lie group right actions $\triangleleft \colon P \times G \to P$ and $\blacktriangleleft \colon P' \times G \to P'$, respectively.*

Then the following are equivalent:

1. *$\Phi \colon P \to P'$ is an id_G-principal bundle isomorphism along id_M between $\pi \colon P \to M$ and $\pi' \colon P' \to M$.*
2. *$\Phi \colon P \to P'$ is an isomorphism of fibre bundles with effective structure group action $\bullet \colon G \times G \to G$.*

Proof 4.5. Suppose first that $\Phi \colon P \to P'$ is a id_G-principal bundle isomorphism along id_M. Let $\alpha \in \mathcal{B}$ and $\alpha' \in \mathcal{B}'$ be principal bundle charts of $\pi \colon P \to M$ and $\pi' \colon P' \to M$, respectively, with the property that

$\pi[\mathrm{Dom}(\alpha)] \cap \pi'[\mathrm{Dom}(\alpha')] \neq \emptyset$. Then for every $p \in \pi[\mathrm{Dom}(\alpha)] \cap \pi'[\mathrm{Dom}(\alpha')]$ and every $g \in G$, it holds that:

$$\begin{aligned} \alpha' \circ \Phi \circ (\pi, \alpha)^{-1}(p, g) &= \alpha' \circ \Phi \circ (\pi, \alpha)^{-1}(p, e \bullet g), \\ &= \alpha' \circ \Phi \left((\pi, \alpha)^{-1}(p, e) \triangleleft g\right), \\ &= \alpha' \left(\Phi \circ (\pi, \alpha)^{-1}(p, e) \blacktriangleleft g\right), \\ &= \alpha' \circ \Phi \circ (\pi, \alpha)^{-1}(p, e) \bullet g. \end{aligned}$$

Here we used that the maps $\alpha\colon \mathrm{Dom}(\alpha) \to G$, $\alpha'\colon \mathrm{Dom}(\alpha') \to G$ and $\Phi\colon P \to P'$ are id_G-equivariant. It is left to observe that the map

$$\tau_{\alpha'\alpha}\colon \pi[\mathrm{Dom}(\alpha)] \cap \pi'[\mathrm{Dom}(\alpha')] \to G, \quad p \mapsto \alpha' \circ \Phi \circ (\pi, \alpha)^{-1}(p, e) \tag{4.28}$$

is smooth.

Now suppose instead that $\Phi\colon P \to P'$ is an isomorphism of fibre bundles with effective structure group action $\bullet\colon G \times G \to G$. Let $a \in P$ and $g \in G$. There exist principal bundle charts α and α' of $\pi\colon P \to M$ and $\pi'\colon P' \to M$, respectively, such that $\pi(a) \in \pi[\mathrm{Dom}(\alpha)] \cap \pi'[\mathrm{Dom}(\alpha')]$. Then:

$$\begin{aligned} \Phi(a \triangleleft g) &= (\pi', \alpha')^{-1} \circ (\pi', \alpha') \circ \Phi \circ (\pi, \alpha)^{-1} \left(\pi(a), \alpha(a \triangleleft g)\right), \\ &= (\pi', \alpha')^{-1} \circ (\pi', \alpha') \circ \Phi \circ (\pi, \alpha)^{-1} \left(\pi(a), \alpha(a) \bullet g\right), \\ &= (\pi', \alpha')^{-1} \left(\pi(a), \tau_{\alpha'\alpha}\left(\pi(a)\right) \bullet \alpha(a) \bullet g\right), \\ &= \Phi(a) \blacktriangleleft g. \end{aligned}$$

We used that $\alpha\colon \mathrm{Dom}(\alpha) \to G$ and $\alpha'\colon \mathrm{Dom}(\alpha') \to G$ are id_G-equivariant and that $\Phi\colon P \to P'$ satisfies equation (3.4) and preserves fibres, i.e., $\pi' \circ \Phi = \pi$. The above equation tells us that Φ is an id_G-principal bundle homomorphism along id_M. Applying the same reasoning to the inverse Φ^{-1} concludes the proof.

Corollary 4.6. *Two principal bundles $\pi\colon P \to M$ and $\pi'\colon P' \to M$ over M with respect to Lie group right actions $\triangleleft\colon P \times G \to P$ and $\blacktriangleleft\colon P' \times G \to P'$, respectively, that are id_G-principal bundle isomorphic along id_M are also isomorphic as fibre bundles with effective structure group action $\bullet\colon G \times G \to G$.*

This proves consistency of the two concepts of bundle isomorphism.

Lemma 4.7. *The Lie group right action $\triangleleft\colon P \times G \to P$ of a principal bundle $\pi\colon P \to M$ acts transitively on the fibres of $\pi\colon P \to M$, i.e.,*

$$\forall a, b \in P\colon\ [\pi(a) = \pi(b) \implies \exists g \in G\colon b = a \triangleleft g]\,. \tag{4.29}$$

Proof 4.7. Let $a, b \in P$ such that $\pi(a) = \pi(b)$. Let $\alpha \in \mathcal{B}$ be a principal bundle chart of $\pi\colon P \to M$ at $\pi(a)$. The typical fibre of $\pi\colon P \to M$ is G. Consequently, we have $\alpha(a), \alpha(b) \in G$. Since G is a group, there exists $g \in G$ such that $\alpha(b) = \alpha(a) \bullet g$. Using the fact that α is id_G-equivariant, permits us to deduce that

$$\alpha(b) = \alpha(a) \bullet g = \alpha(a \triangleleft g). \tag{4.30}$$

It follows that $b = a \triangleleft g$ due to the fact that a and b lie in the same fibre.

Lemma 4.8. *Let $\triangleleft\colon M \times G \to M$ be a free and transitive Lie group right action.*

Then M is diffeomorphic to G.

Proof 4.8. Let $p \in M$. We claim that the smooth map

$$p\triangleleft\colon G \to M, \quad g \mapsto p \triangleleft g \tag{4.31}$$

is a diffeomorphism. We will show that by demonstrating that $p\triangleleft\colon G \to M$ is a bijective constant rank map. The constant rank theorem and the inverse function theorem then ensure that $p\triangleleft\colon G \to M$ is a diffeomorphism.

First note that $p\triangleleft\colon G \to M$ is indeed a bijection. It is surjective since $\triangleleft$ is transitive, and injective since $\triangleleft$ is free. Now suppose $g, h \in G$. Then, $\triangleleft\colon M \times G \to M$ being a right action, it holds that

$$\left(p \triangleleft \left(g \bullet h^{-1}\right)\right) \triangleleft h = p \triangleleft g. \tag{4.32}$$

In other words, for all $h \in G$ it holds that

$$p\triangleleft = (\triangleleft h) \circ (p\triangleleft) \circ \left(\bullet h^{-1}\right). \tag{4.33}$$

Taking the differential $(p\triangleleft)_{*,h}$ at h yields

$$(p\triangleleft)_{*,h} = (\triangleleft h)_{*,p} \circ (p\triangleleft)_{*,e} \circ \left(\bullet h^{-1}\right)_{*,h}. \tag{4.34}$$

Recall that $\triangleleft h\colon M \to M$ and $\triangleleft h^{-1}\colon M \to M$ are smooth maps and mutual inverses. Hence $\triangleleft h\colon M \to M$ is a diffeomorphism. Analogously, $\bullet h^{-1}\colon G \to G$ is a diffeomorphism.

This proves that the rank of $(p\triangleleft)\colon G \to M$ is constant.

Corollary 4.9. *Let $\pi\colon P \to M$ be a principal bundle with respect to the Lie group right action $\triangleleft\colon P \times G \to P$. Then the fibres of $\pi\colon P \to M$ are isomorphic to the Lie group G.*

Proof 4.9. Let $p \in M$ and denote by P_p the fibre $\pi^{-1}[\{p\}]$ over p. By item 3 of definition 4.2, the Lie group right action $\triangleleft$ is fibre-preserving. Recall as well that the fibre P_p is an embedded submanifold of P. Hence we can define the restricted Lie group right action $\triangleleft_p\colon P_p \times G \to P_p$ acting on the fibre. By item 1 of definition 4.2, the restricted Lie group right action $\triangleleft_p$ acts freely on P_p. Furthermore, lemma 4.7 showed that $\triangleleft_p$ acts transitively on P_p. Lemma 4.8 then states that P_p is diffeomorphic to G.

Lemma 4.10. *Let $\pi\colon P \to M$ be a principal bundle with respect to the Lie group right action $\triangleleft\colon P \times G \to P$ and let $\psi\colon U \to P$ be a local section.*

Then the smooth map

$$\Psi\colon U \times G \to \pi^{-1}[U], \quad (q, g) \mapsto \psi(q) \triangleleft g \tag{4.35}$$

is a diffeomorphism with the property that its inverse $\Phi^{-1} = (\pi, \alpha)$ gives rise to a principal bundle chart $\alpha\colon \pi^{-1}[U] \to G$.

Proof 4.10. The smooth map $\Psi\colon U \times G \to \pi^{-1}[U]$ is a bijection due to the following two facts. First, $\triangleleft$ acts freely and transitively on the fibres. Second, the local section $\Psi\colon U \to P$ hits each fibre over U exactly once.

Now let $(p, g) \in U \times G$. Suppose $(X, A) \in T_pU \times T_gG$ is such that

$$\Psi_{*,(p,g)}(X, A) = 0. \tag{4.36}$$

Applying $\pi_{*,\Psi(p,g)}$ to the above expression yields

$$0 = \pi_{*,\Psi(p,g)} \circ \Psi_{*,(p,g)}(X, A) = (\pi \circ \Psi)_{*,(p,g)}(X, A). \tag{4.37}$$

But $\pi \circ \Psi = \mathrm{pr}_1$, so we conclude $X = 0$.

Furthermore, since $\psi(p)\triangleleft\colon G \to P_p, g \mapsto \Phi(p, g)$ is a diffeomorphism (cf. corollary 4.9), $A = 0$ follows as well. We conclude that Ψ has full rank at (p, g). The choice of $(p, g) \in U \times G$ was arbitrary. Therefore Ψ is not only a bijection but has constant rank, and as such, is a diffeomorphism, as guaranteed by the constant rank theorem and the inverse function theorem.

Now observe that the inverse map $\Psi^{-1}\colon \pi^{-1}[U] \times U \times G$ satisfies $\mathrm{pr}_1 \circ \Psi^{-1} = \pi$. Consequently, we can write $\Psi^{-1} = (\pi, \alpha)$ where $\alpha\colon \pi^{-1}[U] \to G$ is a smooth map. We already showed that α is a bundle chart. It is left to show that α is id_G-equivariant and thus qualifies as a principal bundle chart. This is most easily done by direct inspection of Ψ. Let $p \in M$ and $g, h \in G$. Then:

$$\Psi(p, h \bullet g) = \psi(p) \triangleleft (h \bullet g) = (\psi(p) \triangleleft h) \triangleleft g = \Psi(p, h) \triangleleft g. \tag{4.38}$$

The result follows by applying α to both sides and letting $h = \alpha(a)$ for some $a \in P_p$.

Corollary 4.11. *A principal bundle is trivial if and only if it admits a global section.*

Definition 4.7 (Parallelizable manifold)**.** We say that a smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ is **parallelizable** if its frame bundle $\hat{\pi}\colon \mathrm{Fr}(TM) \to M$ is trivial. By corollary 4.11, this is the case if and only if there exists a global frame for $\pi\colon TM \to M$, thus the name **parallelizable**.

Lemma 4.12. *Let $\pi\colon P \to M$ and $\pi'\colon P' \to M'$ be two principal bundles with respect to Lie group right actions $\triangleleft\colon P \times G \to P$ and $\blacktriangleleft\colon P' \times G' \to P'$, respectively. Suppose we are given a diffeomorphism $\varphi\colon M \to M'$, a Lie group isomorphism $\rho\colon G \to G'$ and a ρ-principal bundle morphism Φ along φ from $\pi\colon P \to M$ to $\pi'\colon P' \to M'$.*

Then Φ is a ρ-principal bundle isomorphism along φ between $\pi\colon P \to M$ and $\pi'\colon P' \to M'$.

Proof 4.12. Note that if either $P = \emptyset$ or $P' = \emptyset$, then $P = M = M' = P' = \emptyset$, and the claim is vacuously true. It suffices to note that P is empty if and only if M is empty, knowing that $\pi\colon P \to M$ is a surjection. Analogously, P' is empty if and only if M' is empty. Finally, M is empty if and only if M' is empty due to $\varphi\colon M \to M'$ being a diffeomorphism.

Subproof ($\Phi\colon P \to P'$ is injective). Let $a, b \in P$ such that $\Phi(a) = \Phi(b)$. Then also $\pi' \circ \Phi(a) = \pi' \circ \Phi(b)$. Using the fact that Φ is a bundle morphism along φ, it follows that $\varphi \circ \pi(a) = \varphi \circ \pi(b)$. In turn, since $\varphi\colon M \to M'$ is a diffeomorphism, we find that $\pi(a) = \pi(b)$, i.e., a and b lie in the same fibre of $\pi\colon P \to M$.

Recall that $\pi\colon P \to M$ is a principal bundle with Lie group right action $\triangleleft\colon P \times G \to P$. By lemma 4.7, $\triangleleft$ acts transitively on the fibres of $\pi\colon P \to M$. That is, there exists an element $g \in G$ that relates a and b through $b = a \triangleleft g$.

Using the fact that $\Phi\colon P \to P'$ is ρ-equivariant, we arrive at the conclusion that

$$\Phi(a) = \Phi(b) = \Phi(a \triangleleft g) = \Phi(a) \blacktriangleleft \rho(g). \tag{4.39}$$

Since $\blacktriangleleft$ is free, it follows that $\rho(g) = e_{G'}$. This is only possible if $g = e_G$, due to the fact that $\rho\colon G \to G'$ is a Lie group isomorphism. Finally, $b = a \triangleleft e_G = a$ follows from the definition of a right action.

Subproof ($\Phi\colon P \to P'$ is surjective). Let $a' \in P'$. Since $\varphi\colon M \to M'$ is a diffeomorphism we can compute an element $\varphi^{-1} \circ \pi'(a')$ in M. Choose an element $b \in P$ (recall that $\pi\colon P \to M$ is a surjection) such that $\pi(b) = \varphi^{-1} \circ \pi'(a')$. Then $\Phi(b)$ and a' lie in the same fibre of $\pi'\colon P' \to M'$:

$$\pi' \circ \Phi(b) = \varphi \circ \pi(b) = \pi'(a). \tag{4.40}$$

Since $\blacktriangleleft\colon P' \times G' \to P'$ acts freely and, as we have seen in lemma 4.7, transitively on the fibres, there exists a unique $g' \in G'$ that relates the two elements a' and $\Phi(b)$ in the fibre ${\pi'}^{-1}[\{\pi'(a')\}]$ according to $a' = \Phi(b) \blacktriangleleft g'$. As $\Phi\colon P \to P'$ is ρ-equivariant, we obtain:

$$a' = \Phi(b) \blacktriangleleft g' = \Phi\left(b \triangleleft \rho^{-1}(g')\right). \tag{4.41}$$

We thus found an element $a = b \triangleleft \rho^{-1}(g') \in P$ with the property that $a' = \Phi(a)$.

Subproof ($\Phi^{-1}\colon P' \to P$ is ρ^{-1}-equivariant). Let $a' \in P'$ and $g' \in G'$. Then,

$$\Phi^{-1}\left(a' \blacktriangleleft g'\right) = \Phi^{-1}\left(\Phi(\Phi^{-1}(a')) \blacktriangleleft \rho\left(\rho^{-1}(g')\right)\right), \tag{4.42}$$

since both $\Phi\colon P \to P'$ and $\rho\colon G \to G'$ are bijections. Using the fact that Φ is ρ-equivariant then yields

$$= \Phi^{-1}\left(\Phi\left(\Phi^{-1}(a') \triangleleft \rho^{-1}(g')\right)\right), \tag{4.43}$$

$$= \Phi^{-1}(a') \triangleleft \rho^{-1}(g'). \tag{4.44}$$

We conclude that $\Phi^{-1}\colon P' \to P$ is ρ^{-1}-equivariant, as desired.

Subproof ($\Phi\colon P \to P'$ is a diffeomorphism). Let p be a point in M, U a neighbourhood of p and $\psi\colon U \to P$ a local section of $\pi\colon P \to M$. Define the smooth maps

$$\Psi\colon U \times G \to \pi^{-1}[U], \quad (q,g) \mapsto \psi(q) \triangleleft g, \tag{4.45}$$

and

$$\Psi'\colon \varphi[U] \times G' \to \pi'^{-1}\left[\varphi[U]\right], \quad (q',g') \mapsto \left(\Phi \circ \psi \circ \varphi^{-1}\right)(q') \blacktriangleleft g'. \tag{4.46}$$

The subject of lemma 4.10 was to show that $\Psi\colon U \times G \to \pi^{-1}[U]$ and $\Psi^{-1}\colon \varphi[U] \times G' \to \pi'^{-1}\left[\varphi[U]\right]$ are local trivializations of $\pi\colon P \to M$ and $\pi'\colon P' \to M'$, respectively.

Now let $q \in M$ and $g \in G$. Then:

$$\left(\Psi'^{-1} \circ \Phi \circ \Psi\right)(q,g) = \Psi'^{-1}\left(\psi(q) \triangleleft g\right), \tag{4.47}$$

$$= \Psi'^{-1}\left(\Phi\left(\psi(q)\right) \blacktriangleleft \rho(g)\right), \tag{4.48}$$

$$= \left(\varphi(q), \rho(g)\right). \tag{4.49}$$

Here we used that Φ is ρ-equivariant. We observe that $\Phi\colon P \to P'$ has constant rank, $\varphi\colon M \to M'$ and $\rho\colon G \to G'$ being diffeomorphisms. As a bijective constant rank map $\Phi\colon P \to P'$ is a diffeomorphism. This concludes the proof.

Proposition 4.13. *Let $\pi_1\colon P_1 \to M$ and $\pi_2\colon P_2 \to M$ be two principal bundles over M with respect to Lie group right actions $\triangleleft\colon P_1 \times G \to P_1$ and $\blacktriangleleft\colon P_2 \times G \to P_2$, respectively.*

Let $\mathcal{I}$ be such that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M with the property that there exist principal bundle atlases $\mathcal{B}_1 = \left\{\alpha_i^1\colon {\pi_1}^{-1}\left[U_i\right] \to U_i \times F \mid i \in \mathcal{I}\right\}$ and $\mathcal{B}_2 = \left\{\alpha_i^2\colon {\pi_2}^{-1}\left[U_i\right] \to U_i \times F \mid i \in \mathcal{I}\right\}$ for $\pi_1\colon P_1 \to M$ and $\pi_2\colon P_2 \to M$, respectively.

Then $\pi_1\colon P_1 \to M$ and $\pi_2\colon P_2 \to M$ are id_G-principal bundle isomorphic along id_M if and only if there exists a family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$ that relates the transition functions $\{\rho_{ij}^1\colon U_i \cap U_j \to G \mid i,j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$ of $\mathcal{B}_1$ to the transition functions $\{\rho_{ij}^2\colon U_i \cap U_j \to G \mid i,j \in \mathcal{I}\colon U_i \cap U_j \neq \emptyset\}$ of $\mathcal{B}_2$ through

$$\forall i,j \in \mathcal{I}\colon \forall p \in U_i \cap U_j\colon \quad \nu_i(p) \bullet \rho_{ij}^1(p) = \rho_{ij}^2(p) \bullet \nu_j(p). \tag{4.50}$$

Proof 4.13. We start off by proving the existence of an id_G-principal bundle isomorphism along id_M between $\pi\colon P_1 \to M$ and $\pi\colon P_2 \to M$ provided the existence of the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$.

Subproof (Constructing a principal bundle isomorphism). We heavily rely on the first part of the proof of proposition 3.2. We proved that

$$\Phi\colon E_1 \to E_2, \quad a \mapsto \left(\pi_2, \alpha_i^2\right)^{-1}\left(\pi_1(a), \nu_i\left(\pi_1(a)\right) \bullet \alpha_i^1(a)\right) \text{ for } i \in \mathcal{I}\colon \pi_1(a) \in U_i \tag{4.51}$$

is an isomorphism of fibre bundles with effective structure group action $\bullet\colon G \times G \to G$ and consequently a id_G-principal bundle isomorphism along id_M between $\pi_1\colon P_1 \to M$ and $\pi_2\colon P_2 \to M$, due to lemma 4.5.

Subproof (Retrieving the family of smooth maps $\{\nu_i\colon U_i \to G \mid i \in \mathcal{I}\}$). Let us now prove the second part of the proposition. Suppose we are given an id_G-principal bundle isomorphism $\Phi\colon P_1 \to P_2$ along id_M between $\pi_1\colon P_1 \to M$ and $\pi_2\colon P_2 \to M$.

It suffices to note that $\Phi\colon P_1 \to P_2$ is an isomorphism of fibre bundles with effective structure group action $\bullet\colon G \times G \to G$ by lemma 4.5. We can then invoke the second part of the proof of proposition 3.2.

Theorem 4.14 (Principal bundle construction theorem). *Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ be a non-empty smooth manifold and $(G, \mathcal{O}_G, \mathcal{A}_G, \bullet)$ a Lie group.*

Suppose we are given $\mathcal{I}$ such that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M and $\{\rho_{ij} : U_i \cap U_j \to G \mid i, j \in \mathcal{I} : U_i \cap U_j \neq \emptyset\}$ is a collection of smooth maps satisfying the following cocycle conditions:

$$\forall i \in \mathcal{I} : \forall p \in U_i : \qquad \rho_{ii}(p) = e_G, \tag{4.52}$$

$$\forall i, j, k \in \mathcal{I} : \ [U_i \cap U_j \cap U_k \neq \emptyset \implies \forall p \in U_i \cap U_j \cap U_k : \qquad \rho_{ik}(p) = \rho_{ij}(p) \bullet \rho_{jk}(p)] \,. \tag{4.53}$$

Then there exists a principal bundle $\pi : P \to M$ over M with respect to a Lie group right action $\triangleleft : P \times G \to P$ and a principal bundle atlas $\mathcal{B} = \{\alpha_i : \pi^{-1}[U_i] \to F \mid i \in \mathcal{I}\}$ whose transition functions agree with the set

$$\{\rho_{ij}\bullet : U_i \cap U_j \to \mathrm{Diff}(F), p \mapsto \rho_{ij}(p)\bullet \mid i, j \in \mathcal{I} : U_i \cap U_j \neq \emptyset\} \,. \tag{4.54}$$

Moreover, by proposition 4.13, $\pi : P \to M$ is unique up to principal bundle isomorphism.

Proof 4.14. Let us review the constructive part of the proof of theorem 3.3.

We start off with the disjoint union of the patches $U_i \times G$

$$\mathcal{P} := \bigcup \{U_i \times G \times \{i\} \mid i \in \mathcal{I}\} \subseteq M \times G \times \mathcal{I} \tag{4.55}$$

as underlying set on which we establish an equivalence relation $\sim$ according to

$$\forall (p, g, i), (q, h, j) \in \mathcal{E} : \ [(p, g, i) \sim (q, h, j) :\iff \ p = q \wedge g = \rho_{ij}(p) \bullet h] \,. \tag{4.56}$$

The quotient set $P := \mathcal{P}/_\sim$ serves as underlying set for the total space of the fibre bundle to be constructed. It turns out that both the projection

$$\pi : P \to M, \quad [(p, g, i)]_\sim \to p \tag{4.57}$$

and the bundle atlas

$$\mathcal{B} := \{\alpha_i : \pi^{-1}[U_i] \to G, \ [(p, g, i)]_\sim \mapsto g \mid i \in \mathcal{I}\} \tag{4.58}$$

are well-defined. Finally, the projection $\pi : P \to M$ together with the bundle atlas $\mathcal{B}$ is a fibre bundle with effective structure group action $\bullet : G \times G \to G$. This was the subject of theorem 3.3.

It is now left to show that $\pi : P \to M$ is also canonically a principal bundle. To this end we will canonically introduce a Lie group right action $\triangleleft : P \times G \to P$ and then show that $\mathcal{B}$ is a principal bundle atlas. Define

$$\triangleleft : P \times G \to P, \quad ([(p, h, i)]_\sim, g) \mapsto [(p, h \bullet g, i)]_\sim \,. \tag{4.59}$$

It is straightforward to check that $\triangleleft$ is well-defined and indeed a right action. It is also manifestly fibre-preserving. Let us check that it is also free. Suppose there exist $[(p, g, i)]_\sim \in P$ and $h \in G$ such that $[(p, g, i)]_\sim \triangleleft h = [(p, g, i)]_\sim$. But then $h = e$. This proves that $\triangleleft$ is free.

It is left to show that $\mathcal{B}$ is a principal bundle atlas. Let $i \in \mathcal{I}$ and recall the definition of the bundle chart

$$\alpha_i : \pi^{-1}[U_i] \to G, \quad [(p, g, i)]_\sim \mapsto g. \tag{4.60}$$

We read off that α_i is id_G-equivariant with respect to the action $\triangleleft$ of G on P.

Example 4.2. Suppose we are given two principal bundles $\pi : P \to M$ and $\pi' : P' \to M$ with respect to Lie group right actions $\triangleleft : P \times G \to P$ and $\blacktriangleleft : P' \times G' \to P'$, respectively. Let $\mathcal{I}$ be a set indexing an open cover $\{U_i \mid i \in \mathcal{I}\}$ of M and principal bundle atlases $\{\alpha_i : \pi^{-1}[U_i] \to G \mid i \in \mathcal{I}\}$ and $\{\alpha'_i : \pi'^{-1}[U_i] \to G' \mid i \in \mathcal{I}\}$ for $\pi : P \to M$ and $\pi' : P' \to M$, respectively. Recall that by proposition 3.1, the two collections of bundle chart transition maps $\{\rho_{ij} : U_i \cap U_j \to G \mid i, j \in \mathcal{I} : U_i \cap U_j \neq \emptyset\}$ and $\{\rho'_{ij} : U_i \cap U_j \to G' \mid i, j \in \mathcal{I} : U_i \cap U_j \neq \emptyset\}$ satisfy the cocycle conditions (3.2) and (3.3).

Define

$$\overset{\times}{\rho}_{ij} : U_i \cap U_j \to G \times G', \quad p \mapsto \left(\rho_{ij}(p), {\rho'}_{ij}(p)\right) \,. \tag{4.61}$$

It is straightforward to check that the collection $\{{\rho^\times}_{ij} \mid i, j \in \mathcal{I} : U_i \cap U_j \neq \emptyset\}$ satisfies the cocycle conditions as well, and, is smooth. Using the Principal bundle construction theorem, we can thus construct a principal bundle $\Pi : P \boxtimes P' \to M$ with structure group $G \times G'$ in a canonical way.

5 Associated bundles

Definition 5.1. Let $\pi \colon E \to M$ and $\pi' \colon E' \to M$ be fibre bundles with effective structure group actions $\triangleright \colon G \times F \to F$ and $\blacktriangleright \colon G \times F' \to F'$, respectively.

Let $\mathcal{I}$ be such that $\{U_i \mid i \in \mathcal{I}\}$ is an open cover of M with the property that there exist a $(G, \triangleright)$-compatible bundle atlas $\mathcal{B} = \left\{\alpha_i \colon \pi^{-1}[U_i] \to F \mid i \in \mathcal{I}\right\}$ for $\pi \colon E \to M$ and a $(G, \blacktriangleright)$-compatible bundle atlas $\mathcal{B}' = \{\alpha'_i \colon {\pi'}^{-1}[U_i] \to F \mid i \in \mathcal{I}\}$ for $\pi' \colon E' \to M$.

We say that $\pi \colon E \to M$ and $\pi' \colon E' \to M$ are **associated** if there exists a family of smooth maps $\{\nu_i \colon U_i \to G \mid i \in \mathcal{I}\}$ that relates the G-valued transition functions $\{\rho_{ij} \colon U_i \cap U_j \to G \mid i, j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\}$ of $\mathcal{B}_1$ to the transition functions $\left\{{\rho'}_{ij} \colon U_i \cap U_j \to G \mid i, j \in \mathcal{I} \colon U_i \cap U_j \neq \emptyset\right\}$ of $\mathcal{B}_2$ through:

$$\forall i, j \in \mathcal{I} \colon \forall p \in U_i \cap U_j \colon \quad \nu_i(p) \bullet \rho_{ij}(p) = {\rho'}_{ij}(p) \bullet \nu_j(p). \tag{5.1}$$

Remark 5.1. Recall that by an adequate choice of $\mathcal{B}$ and $\mathcal{B}'$ for the associated bundles $\pi \colon E \to M$ and $\pi' \colon E' \to M$, the G-valued transition functions can always be made coincide. See remark 3.4.

Definition 5.2. Let $\pi \colon E \to M$ be a fibre bundle with effective structure group action $\triangleright \colon G \times F \to F$ and let $\mathcal{B}$ be a $(G, \triangleright)$-compatible bundle atlas for $\pi \colon E \to M$. Starting from the G-valued transition functions of $\mathcal{B}$ we can build a unique principal bundle $\hat{\pi} \colon P \to M$ up to principal bundle isomorphism, by means of theorem 4.14. Any principal bundle $\hat{\pi} \colon P \to M$ from this isomorphism class is said to be **associated** to $\pi \colon E \to M$.

Definition 5.3. Let $\pi \colon P \to M$ be a principal bundle with respect to a Lie group right action $\triangleleft \colon P \times G \to P$. Denote by $\mathcal{B}$ a principal bundle atlas for $\pi \colon P \to M$. By lemma 4.1, $\mathcal{B}$ is in particular $(G, \bullet)$-compatible. Suppose we are given an effective Lie group left action $\triangleright \colon G \times F \to F$.

Starting from the G-valued transition functions of $\mathcal{B}$ we can build a unique fibre bundle $\tilde{\pi} \colon E \to M$ with effective structure group action $\triangleright \colon G \times F \to F$ up to isomorphism of fibre bundles with effective structure group action $\triangleright \colon G \times F \to F$, by means of theorem 3.3. Any fibre bundle $\tilde{\pi} \colon E \to M$ with effective structure group action $\triangleright \colon G \times F \to F$ from this isomorphism class is said to be **associated** to $\pi \colon P \to M$ via $\triangleright$.

Example 5.1. Let $\pi \colon P \to M$ be a principal bundle with respect to a Lie group right action $\triangleleft \colon P \times G \to P$ and suppose we are given an effective Lie group left action $\triangleright \colon G \times F \to F$ and a principal bundle atlas $\mathcal{B}$.

We will now provide another concrete implementation of the fibre bundle with effective structure group action $\triangleright$ that is associated to $\pi \colon P \to M$.

To this end, we first define an equivalence relation on the Cartesian product $P \times F$ according to

$$\forall (a_1, f_1), (a_2, f_2) \in P \times F \colon \quad \left[(a_1, f_1) \sim (a_2, f_2) \;:\Longleftrightarrow\; \exists g \in G \colon (a_2, f_2) = (a_1 \triangleleft g, g^{-1} \triangleright f_1)\right]. \tag{5.2}$$

Denote by $P \triangleright F := (P \times F)/_\sim$ the quotient set and by $[a, f]$ the equivalence class of an element $(a, f) \in P \times F$. Define the projection

$$\overset{\triangleright}{\pi} \colon P \triangleright F \to M, \quad [a, f] \mapsto \pi(a), \tag{5.3}$$

which is well-defined since $\pi \circ \mathrm{pr}_1$ is constant on the equivalence classes. Define a bundle atlas

$$\overset{\triangleright}{\mathcal{B}} := \left\{ \overset{\triangleright}{\alpha} \colon \overset{\triangleright}{\pi}^{-1}[\pi[\mathrm{Dom}(\alpha)]] \to F, [a, f] \mapsto \alpha(a) \triangleright f \;\middle|\; \alpha \in \mathcal{B} \right\}, \tag{5.4}$$

which itself is well-defined due to the choice of the equivalence relation $\sim$ together with the fact that each principal bundle chart α is id_G-equivariant.

Using theorem 2.1, we can show that $\pi^{\triangleright} \colon P \triangleright F \to M$ is a fibre bundle. The bundle atlas $\mathcal{B}^{\triangleright}$ is manifestly $(G, \triangleright)$-compatible and its G-valued transition functions match those of $\mathcal{B}$. Therefore, $\pi^{\triangleright} \colon P \triangleright F \to M$ is associated to $\pi \colon P \to M$ via $\triangleright$.

Remark 5.2. The concept of associated bundles formalizes the idea that physicists have in mind when then they define, for instance, covariant vector fields, also known as 1-forms. Physicists like to stress that "a contravariant vector field is an k-component entity whose components transform under a local linear transformation A according to the formula $w \mapsto \big((A^{-1})^m{}_1 w_m, \ldots, (A^{-1})^m{}_k w_m\big)$". That is, physicists think entirely in terms of the transformation behaviour of components. What physicists don't bother to point out is that they are implicitly working with a chosen frame, or in more general terms, a chosen principal bundle chart, in relation to which the components are to be understood.

Remark 5.3. The frame bundle $\hat{\pi} \colon \mathrm{Fr}(E) \to M$ of a vector bundle $\pi \colon E \to M$ is the principal bundle associated to $\pi \colon E \to M$.

Remark 5.4. The Fibre bundle construction theorem and Principal bundle construction theorem enable us to construct a wide variety of new principal bundles and associated bundles thereof. The next section will provide us with some examples.

6 Constructing more vector bundles

This section will treat about how to make rigorous definitions such as tensor fields.

Example 6.1. Let $\pi\colon E \to M$ be a vector bundle of rank k. Recall that its typical fibre is $\mathbb{R}^k$ which is acted upon by the general linear group $\mathrm{GL}(k,\mathbb{R})$ by means of its defining representation

$$\triangleright\colon\ \mathrm{GL}(k,\mathbb{R}) \times \mathbb{R}^k \to \mathbb{R}^k, \quad (A, v) \mapsto \left(A^1{}_m v^m, \ldots, A^k{}_m v^m\right). \tag{6.1}$$

Making use of the Fibre bundle construction theorem, we can simply define the **dual bundle** $\pi^*\colon E^* \to M$ as the associated bundle via the Lie group left action

$$\blacktriangleright\colon\ \mathrm{GL}(k,\mathbb{R}) \times \mathbb{R}^k \to \mathbb{R}^k, \quad (A, w) \mapsto \left(A_1{}^m w_m, \ldots, A_k{}^m w_m\right), \tag{6.2}$$

where $A_n{}^m := \left(A^{-1}\right)^m{}_n$ denotes the components of the inverse group element.

Alternatively, we can directly construct it using the low-level theorem 2.1. Having in mind remark 5.2, we might regard this as an optional step. Yet, from the conceptual point of view, it may be worthwhile to explicitly construct the bundle. We may want to utilize the collection of dual spaces of the fibres

$$E^* := \{E_p{}^* \mid p \in M\} \tag{6.3}$$

as a total space for the dual bundle, together with the projection

$$\pi^*\colon E^* \to M, \quad \omega \mapsto p \text{ such that } \omega \in E_p{}^*. \tag{6.4}$$

Let α be a vector bundle chart of $\pi\colon E \to M$ and define for $1 \le i \le k$ the vector fields

$$e_i\colon \pi[\mathrm{Dom}(\alpha)] \to E, \quad q \mapsto (\pi, \alpha)^{-1}(q, \hat{e}_i), \tag{6.5}$$

where $(\hat{e}_1, \ldots, \hat{e}_k)$ is the standard basis of $\mathbb{R}^k$. The collection $(e_1, \ldots, e_k)$ is a local frame and we can use it in order to define a bundle chart

$$\alpha^*\colon {\pi^*}^{-1}\left[\pi[\mathrm{Dom}(\alpha)]\right] \to \mathbb{R}^k, \quad \omega \mapsto \left(\omega\left(e_1(\pi^*(\omega))\right), \ldots, \omega\left(e_k(\pi^*(\omega))\right)\right). \tag{6.6}$$

The bundle atlas

$$\mathcal{B}^* := \{\alpha^* \mid \alpha \in \mathcal{B}\} \tag{6.7}$$

satisfies the hypothesis of theorem 2.1 and in particular is composed of linear maps. So $\pi^*\colon E^* \to M$ is a vector bundle.

Example 6.2. Let $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$ be vector bundles of ranks k and l, respectively. And denote by $\hat{\pi}_1\colon \mathrm{Fr}(E_1) \to M$ and $\hat{\pi}_2\colon \mathrm{Fr}(E_2) \to M$ their respective frame bundles. Define a left action of the direct product $\mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(l,\mathbb{R})$ on $\mathbb{R}^{k+l}$ according to

$$\begin{aligned} \triangleright\colon\ & (\mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(l,\mathbb{R})) \times \mathbb{R}^{k+l} \to \mathbb{R}^{k+l}, \\ & ((g,h), v) \mapsto \left(g^1{}_m v^m, \ldots, g^k{}_m v^m, h^1{}_n v^{k+n}, \ldots, h^l{}_n v^{k+n}\right). \end{aligned} \tag{6.8}$$

The **direct sum bundle** $\pi^{\oplus}\colon E_1 \oplus E_2 \to M$ is the associated bundle to the product bundle of the frame bundles by means of the Lie group left action $\triangleright$.

Contrary to the case of the dual bundle, we can also construct the direct sum bundle in an easy and straightforward way. As total space we can use the embedded submanifold

$$E_1 \oplus E_2 := (\pi_1 \times \pi_2)^{-1}\left[\{(p,p) \mid p \in M\}\right] \subseteq E_1 \times E_2. \tag{6.9}$$

The projection

$$\overset{\oplus}{\pi}\colon E_1 \oplus E_2 \to M, \quad (a_1, a_2) \mapsto \pi_1(a_1) \tag{6.10}$$

then is manifestly smooth and a surjection. Given vector bundle charts α_1 and α_2 of $\pi_1\colon E_1 \to M$ and $\pi_2\colon E_2 \to M$, respectively, the map

$$\alpha_1 \times \alpha_2 \colon \overset{\oplus}{\pi}^{-1}\left[\pi_1[\mathrm{Dom}(\alpha_1)] \cap \pi_2[\mathrm{Dom}(\alpha_2)]\right] \to \mathbb{R}^{k+l} \tag{6.11}$$

is smooth and linear. This provides a vector bundles atlas

$$\mathcal{B}_1 \oplus \mathcal{B}_2 := \left\{\alpha_1 \times \alpha_2 \mid \alpha_1 \in \mathcal{B}_1, \alpha_2 \in \mathcal{B}_2 \colon \pi_1[\mathrm{Dom}(\alpha_1)] \cap \pi_2[\mathrm{Dom}(\alpha_2)] \neq \emptyset\right\}. \tag{6.12}$$

By remark 2.1, this suffices to define a vector bundle.

Example 6.3. Let $\pi\colon E \to M$ and $\pi'\colon E' \to M$ be vector bundles of ranks k and l, respectively. Denote by $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ and $\hat{\pi}'\colon \mathrm{Fr}(E') \to M$ their respective frame bundles. Denote by $(e_1, \dots, e_k)$ the standard basis of $\mathbb{R}^k$ and by $(b_1, \dots, b_l)$ the standard basis of $\mathbb{R}^l$. Define a left action of the direct product $\mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(l,\mathbb{R})$ on $\mathbb{R}^k \otimes \mathbb{R}^l$ according to

$$\begin{aligned} \triangleright\colon\ & (\mathrm{GL}(k,\mathbb{R}) \times \mathrm{GL}(l,\mathbb{R})) \times \left(\mathbb{R}^k \otimes \mathbb{R}^l\right) \to \mathbb{R}^k \otimes \mathbb{R}^l, \\ & \left((g,h), v^{ij} e_i \otimes b_j\right) \mapsto g^i{}_m h^j{}_n v^{mn} e_i \otimes b_j. \end{aligned} \tag{6.13}$$

The **tensor product bundle** $\pi^{\otimes}\colon E \otimes E' \to M$ is the associated bundle to the product bundle of the frame bundles by means of the Lie group left action $\triangleright$.

Alternatively, we can explicitly define the total space as as set

$$E \otimes E' := \{E_p \otimes E'{}_p \mid p \in M\} \tag{6.14}$$

with projection

$$\overset{\otimes}{\pi}\colon E \otimes E' \to M, \quad a \mapsto p \text{ such that } a \in E_p \otimes E'{}_p \tag{6.15}$$

and bundle atlas

$$\begin{aligned} \mathcal{B} \otimes \mathcal{B}' := \Big\{ \alpha \otimes \alpha' \colon \overset{\otimes}{\pi}^{-1}\left[\pi\left[\mathrm{Dom}(\alpha)\right] \cap \pi'\left[\mathrm{Dom}(\alpha')\right]\right] \to \mathbb{R}^k \otimes \mathbb{R}^l, a \mapsto \left(\alpha|_{E_{\overset{\otimes}{\pi}(a)}} \otimes \alpha'|_{E'_{\overset{\otimes}{\pi}(a)}}\right)(a) \\ \Big| \alpha \in \mathcal{B}, \alpha' \in \mathcal{B}' \colon \pi[\mathrm{Dom}(\alpha)] \cap \pi'[\mathrm{Dom}(\alpha')] \neq \emptyset \Big\}. \end{aligned} \tag{6.16}$$

Then, after some work, we can invoke theorem 2.1 in order to establish that $\pi^{\otimes}\colon E \otimes E' \to M$ is a vector bundle.

More commonly, we will construct tensor product bundles of vector bundles that are associated to the same frame bundle.

Example 6.4. Let $\pi\colon E \to M$ be a vector bundle of rank k. Denote by $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ its frame bundle, by $(e_1, \dots, e_k)$ the standard basis of $\mathbb{R}^k$ and by $(e^1, \dots, e^k)$ the dual basis of the standard basis.

Define a left action of the direct product $\mathrm{GL}(k,\mathbb{R})$ on $\mathbb{R}^{k^{\otimes(r+s)}}$ according to

$$\begin{aligned}\triangleright\colon\ & \mathrm{GL}(k,\mathbb{R}) \times \mathbb{R}^{k^{\otimes(r+s)}} \to \mathbb{R}^{k^{\otimes(r+s)}}, \\ & \left(A, v^{i_1\cdots i_r}{}_{j_1\cdots j_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}\right) \\ & \mapsto A^{i_1}{}_{m_1} \cdots A^{i_r}{}_{m_r} A_{j_1}{}^{n_1} \cdots A_{j_s}{}^{n_s} v^{m_1\cdots m_r}{}_{n_1\cdots n_s} e_{i_1} \otimes \cdots \otimes e_{i_r} \otimes e^{j_1} \otimes \cdots \otimes e^{j_s}.\end{aligned} \tag{6.17}$$

The (r,s)**-tensor product bundle** $\pi^{(r,s)}\colon E^{(r,s)} \to M$ of $\pi\colon E \to M$ is the associated bundle to the frame bundle $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ by means of the Lie group left action $\triangleright$.

Alternatively we can construct the bundle directly starting form the set

$$E^{(r,s)} := \left\{ E_p{}^{\otimes r} \otimes E_p{}^{*\otimes s} \mid p \in M \right\} \tag{6.18}$$

as total space with the projection

$$\pi^{(r,s)}\colon E^{(r,s)} \to M, \quad a \mapsto p \text{ such that } a \in E_p{}^{\otimes r} \otimes E_p{}^{*\otimes s}. \tag{6.19}$$

Suppose $\alpha \in \mathcal{B}$ is a vector bundle chart of $\pi\colon E \to M$. Then $\alpha^* \in \mathcal{B}^*$ is the corresponding vector bundle chart of the dual bundle $\pi^*\colon E^* \to M$. We can construct the fibre-wise linear map

$$\alpha^{\otimes r} \otimes \alpha^{*\otimes s}\colon {\pi^{(r,s)}}^{-1}\left[\pi[\mathrm{Dom}(\alpha)]\right] \to \mathbb{R}^{k^{\otimes(r+s)}}, \quad a \mapsto \left(\alpha|^{\otimes r}_{E_{\pi^{(r,s)}(a)}} \otimes \alpha^*|^{\otimes s}_{E^*_{\pi^{(r,s)}(a)}} \right)(a) \tag{6.20}$$

with the intention to use it as a vector bundle chart of the vector bundle to be constructed. Indeed, the collection

$$\mathcal{B}^{(r,s)} := \left\{ \alpha^{\otimes r} \otimes \alpha^{*\otimes s} \mid \alpha \in \mathcal{B} \right\} \tag{6.21}$$

satisfies the hypothesis of theorem 2.1. This makes $\pi^{(r,s)}\colon E^{(r,s)} \to M$ a vector bundle.

Definition 6.1. A section $T \in \Gamma(E^{(r,s)})$ of the (r,s)-tensor product bundle of $\pi\colon E \to M$ is called an (r,s)**-tensor field**. Example 6.4 makes the idea of tensor fields rigorous and elucidates their transformation behaviour in a direct way by establishing its relation to the frame bundle $\hat{\pi}\colon \mathrm{Fr}(E) \to M$ of $\pi\colon E \to M$.

Remark 6.1. Recall that exercise 3.1 verifies that set of tensor fields $\Gamma(E^{(r,s)})$ of a vector bundle, together with pointwise addition $\oplus_{\Gamma(E^{(r,s)})}$ and pointwise scalar multiplication $\boxdot_{\Gamma(E^{(r,s)})}$ forms a unital module

$$\left(\Gamma(E^{(r,s)}), \underset{\Gamma(E^{(r,s)})}{\oplus}, \underset{\Gamma(E^{(r,s)})}{\boxdot} \right) \tag{6.22}$$

over the unital commutative ring of the smooth functions $(C^\infty(M), +_{C^\infty(M)}, \cdot_{C^\infty(M)})$ from exercise B.1..

Proposition 6.1. *The unital module of tensor fields* $\Gamma(E^{(r,s)})$ *with respect to pointwise addition* $\oplus_{\Gamma(E^{(r,s)})}$ *and pointwise scalar multiplication* $\boxdot_{\Gamma(E^{(r,s)})}$ *from remark 6.1 is canonically isomorphic to the unital module of $C^\infty(M)$-multilinear maps* $\mathrm{Mult}_{C^\infty(M)}(\Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s}, C^\infty(M))$ *with respect to the addition* $\oplus_{\mathrm{Mult}}$ *and scalar multiplication* $\boxdot_{\mathrm{Mult}}$ *from exercise B.3, i.e.,*

$$\left(\Gamma(E^{(r,s)}), \underset{\Gamma(E^{(r,s)})}{\oplus}, \underset{\Gamma(E^{(r,s)})}{\boxdot} \right) \cong \left(\mathrm{Mult}_{C^\infty(M)}(\Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s}, C^\infty(M)), \underset{\mathrm{Mult}}{\oplus}, \underset{\mathrm{Mult}}{\boxdot} \right), \tag{6.23}$$

as unital modules over the unital commutative ring of the smooth functions $(C^\infty(M), +_{C^\infty(M)}, \cdot_{C^\infty(M)})$ *from exercise B.1.*

Proof 6.1. First, construct a map

$$i\colon \Gamma(E^{(r,s)}) \to \mathrm{Mult}_{C^\infty(M)}(\Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s}, C^\infty(M)), \tag{6.24}$$

where for every tensor field $S \in \Gamma(E^{(r,s)})$ we have a map

$$i(S)\colon \Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s} \to C^\infty(M) \tag{6.25}$$

that is characterized by its action on a collection of sections $Y_1, \ldots, Y_s \in \Gamma(E)$ of $\pi\colon E \to M$ and sections $\omega_1, \ldots, \omega_r \in \Gamma(E^*)$ of the dual bundle $\pi^*\colon E^* \to M$:

$$i(S)(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s)\colon M \to \mathbb{R}, \quad p \mapsto S(p)\left(\omega_1(p), \ldots, \omega_r(p), Y_1(p), \ldots, Y_s(p)\right). \tag{6.26}$$

Indeed, $i(S)(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s) \in C^\infty(M)$ is a smooth function. For, if $\alpha \in \mathcal{B}$ is a vector bundle chart of $\pi\colon E \to M$ (denote by $\alpha^* \in \mathcal{B}^*$ the corresponding vector bundle chart of the dual bundle and by $\alpha^{\otimes r} \otimes \alpha^{*\otimes s} \in \mathcal{B}^{(r,s)}$ the corresponding vector bundle chart of the (r,s)-tensor product bundle), then for every point $p \in \pi[\mathrm{Dom}(\alpha)]$ it holds that

$$\begin{aligned} &S(p)\left(\omega_1(p), \ldots, \omega_r(p), Y_1(p), \ldots, Y_s(p)\right) \\ &\quad = \left(\alpha^{\otimes s} \otimes \alpha^{*\otimes r}\right)(S(p))\left(\alpha^*(\omega_1(p)), \ldots, \alpha^*(\omega_r(p)), \alpha(Y_1(p)), \ldots, \alpha(Y_s(p))\right). \end{aligned} \tag{6.27}$$

However, $\left(\alpha^{\otimes r} \otimes \alpha^{*\otimes s}\right)(S(p)) \in \mathbb{R}^{(r+s)\cdot k}$, $\alpha^*(\omega_m(p)) \in \mathbb{R}^k$ and $\alpha(Y_n(p)) \in \mathbb{R}^k$ depend smoothly on p. This proves that $i(S)(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s) \in C^\infty(M)$ is indeed a smooth function. Having verified this, it is clear by construction that $i(S)$ is a $C^\infty(M)$-multilinear map.

Second, define the map

$$j\colon\ \mathrm{Mult}_{C^\infty(M)}(\Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s}, C^\infty(M)) \to \Gamma(E^{(r,s)}), \tag{6.28}$$

where for every $C^\infty(M)$-multilinear map $L\colon \Gamma(E^*)^{\times r} \times \Gamma(E)^{\times s} \to C^\infty(M)$ we have the map

$$j(L)\colon M \to E^{(r,s)}, \tag{6.29}$$

that for every point $p \in M$ assigns the tensor $j(L)(p) \in {E_p}^{\otimes r} \otimes \cdots \otimes {E_p}^{*\otimes s}$

$$j(L)(p)\colon {E_p}^{*\times r} \times {E_p}^{\times s} \to \mathbb{R}, (\omega_1(p), \ldots, \omega_r(p), Y_1(p), \ldots, Y_s(p)) \mapsto L(\omega_1, \ldots, \omega_r, Y_1, \ldots, Y_s)(p). \tag{6.30}$$

Note that $j(L)(p)$ is well-defined and depends only on p due to the $C^\infty(M)$-linearity of L.

Suppose now that we are given a vector bundle chart $\alpha \in \mathcal{B}$ of $\pi\colon E \to M$ over $U \in \mathcal{O}_M$ with corresponding vector bundle chart $\alpha^* \in \mathcal{B}^*$ of the dual bundle. Denote by $b_1, \ldots, b_k \in \Gamma(E|_U)$ the corresponding local frame and by $b^1, \ldots, b^k \in \Gamma(E^*|_U)$ the its corresponding dual frame. Then:

$$\left(\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_r} \otimes \alpha^{*j_1} \otimes \cdots \otimes \alpha^{*j_s}\right)(j(L)(p)) = L\left(b^{i_1}, \ldots, b^{i_r}, b_{j_1}, \ldots, b_{j_s}\right)(p). \tag{6.31}$$

But this agrees with the value of $j(L)(p)$ with respect to the corresponding vector bundle chart $\alpha^{\otimes r} \otimes \alpha^{*\otimes s} \in \mathcal{B}^{(r,s)}$ of the (r,s)-tensor product bundle, cf. equation (6.21). This proves that $j(L)\colon M \to E^{(r,s)}$ is a smooth map and thus a section of $\pi^{(r,s)}\colon (E^{(r,s)}) \to M$.

It is left to proof that the two maps i and j are $C^\infty(M)$-linear and are mutual inverses. This is straightforward to verify and is left as an exercise to the reader. This concludes the proof.

Definition 6.2. For every pair of tensor product bundles $E^{(m,n)}$ and $E^{(r,s)}$ we can define an operator

$$\otimes\colon \Gamma(E^{(m,n)}) \times \Gamma(E^{(r,s)}) \to \Gamma(E^{(m+r,n+s)}) \tag{6.32}$$

that assigns to each pair of tensor fields $S \in \Gamma(E^{(m,n)})$ and $T \in \Gamma(E^{(r,s)})$ another tensor field $S \otimes T \in \Gamma(E^{(m+r,n+s)})$, called the **tensor product of** S **and** T, given by the map

$$S \otimes T\colon M \to E^{(m+r,n+s)}, \quad p \mapsto S \underset{p}{\otimes} T, \tag{6.33}$$

where $\otimes_p$ is the tensor product[3] over the fibre E_p and its dual E_p^*.

Definition 6.3. For every tensor product bundle $E^{(r,s)}$ with $r, s \geq 1$ and every pair (m, n) of natural numbers satisfying $1 \leq m \leq r$ and $1 \leq n \leq s$, we can define the operator

$$C_n^m\colon \Gamma(E^{(r,s)}) \to \Gamma(E^{(r-1,s-1)}) \tag{6.34}$$

that assigns to each tensor field $S \in \Gamma(E^{(r,s)})$ another tensor field $C_n^m(S) \in \Gamma(E^{(r-1,s-1)})$, called the **(m,n)-contraction of** S, given by

$$C_n^m(S)\colon M \to E^{(r-1,s-1)}, \quad p \mapsto {C_p}_n^m(S_p), \tag{6.35}$$

where ${C_p}_n^m\colon {E_p}^{\otimes r} \otimes E_p^{*\otimes s} \to {E_p}^{\otimes(r-1)} \otimes E_p^{*\otimes(s-1)}$ is the (m,n)-contraction[3] over the fibre E_p and its dual E_p^*.

Remark 6.2. There are at least three different ways of constructing new vector bundles from given ones, two of which are very abstract and, from the physics point of view, not always very insightful.

One of these two abstract methods is the one invoked multiple times throughout this section using the Fibre bundle construction theorem, sometimes in combination with the Principal bundle construction theorem.

The second abstract method is sometimes referred to as "Metatheorem". In loose terms it says that "everything you can do with vector spaces can be done with vector bundles over the same base space M". It relies on some category theoretic considerations and will not be studied here.

The third method is the down-to-earth approach and uses theorem 2.1 without any fancy tricks. This is usually the most insightful approach as it requires us to provide what grows to become a bundle atlas for the fibre bundle. This process reveals all there is to be learnt about the newly constructed fibre bundle and is in general an exercise worthy of our time.

[3] Covered for instance in *Differential Geometry and Lie Groups for Physicists* by M. Fecko, Page 37, Chapter 2, Section 4

7 Bundle metric

Definition 7.1. Let $\pi\colon E \to M$ be a vector bundle of rank k. A **bundle metric** on $\pi\colon E \to M$ is a global $(0,2)$-tensor field $g \in \Gamma(E^{(0,2)})$ (a section of the $(0,2)$-tensor product bundle $\pi^{(0,2)}\colon E^{(0,2)} \to M$ of $\pi\colon E \to M$) such that for every point $p \in M$ the tensor $g_p \in E_p^* \otimes E_p^*$ is **symmetric**, i.e.,

$$\forall X, Y \in E_p\colon \quad g_p(X,Y) = g_p(Y,X) \tag{7.1}$$

and **non-degenerate**, i.e.,

$$\forall X \in E_p\colon \; [\forall Y \in E_p\colon g_p(X,Y) = 0 \implies X = 0]\,. \tag{7.2}$$

Remark 7.1. On a vector bundle $\pi\colon E \to M$ there always exists a (Riemannian) bundle metric.[4]

Remark 7.2. *Sylvester's law of inertia* [Syl52] states that for every point $p \in M$, there exists a basis $(e_1, \ldots, e_k)$ of E_p with corresponding dual basis $(e^1, \ldots, e^k)$ of $E_p{}^*$ such that g_p assumes the form

$$g_p = \sum_{i=1}^{r} e^i \otimes e^i - \sum_{i=r+1}^{r+s} e^i \otimes e^i, \tag{7.3}$$

for natural numbers r, s. Moreover, the natural numbers r and s do not depend on the particular choice of basis $(e_1, \ldots, e_k)$. We say that g_p has **signature** (r,s). Since g_p is non-degenerate, it holds that $r + s = k$.

Observe that the signature of a bundle metric is locally constant, and consequently (proof left as an exercise to the reader), constant on the connected components of M.

Definition 7.2. A bundle metric $g \in \Gamma(E^{(0,2)})$ of signature $(1, k-1)$ is said to be **Lorentzian**.

Definition 7.3. Let g be a bundle metric of signature $(r, k-r)$ on a vector bundle $\pi\colon E \to M$. A local frame $(e_1, \ldots, e_k)$ over $U \in \mathcal{O}_M$ of $\pi\colon E \to M$ is said to be **orthonormal** if for every point $p \in U$ it holds that

$$g_p = \sum_{i=1}^{r} e_p^i \otimes e_p^i - \sum_{i=r+1}^{k} e_p^i \otimes e_p^i. \tag{7.4}$$

Proposition 7.1. *Let g be a bundle metric of signature $(r, k-r)$ on a vector bundle $\pi\colon E \to M$ and let $p \in M$. Then there exists a local orthonormal frame $e_1, \ldots, e_k \in \Gamma(E|_U)$ over a neighbourhood $U \in \mathcal{O}_M$ of p.*

Proof 7.1. Suppose we are given an orthonormal basis $(v_1, \ldots, v_k)$ of E_p with corresponding dual basis $(v^1, \ldots, v^k)$ of $E_p{}^*$ satisfying

$$g_p = \sum_{i=1}^{r} v^i \otimes v^i - \sum_{i=r+1}^{r+s} v^i \otimes v^i. \tag{7.5}$$

[4] *Riemannian Geometry and Geometric Analysis* (6th edition) by J. Jost, Theorem 2.1.4

Such a basis exists due to *Sylvester's law of inertia*. For every $1 \leq i \leq k$ define the number $\epsilon_i = g_p(v_i, v_i)$. Using some vector bundle chart $\alpha \in \mathcal{B}$ of $\pi \colon E \to M$ at p, we can define for every $1 \leq i \leq k$ the local section

$$b_i \colon \pi[\mathrm{Dom}(\alpha)] \to E, \quad q \mapsto (\pi, \alpha)^{-1} (q, \alpha(v_i)) . \tag{7.6}$$

Due to the fact that for every $q \in \pi[\mathrm{Dom}(\alpha)]$ the map $\alpha|_{E_q} \colon E_q \to \mathbb{R}^k$ is a linear isomorphism, the tuple $b_1, \ldots, b_k$ forms a local frame of $\pi \colon E \to M$ which agrees with $v_1, \ldots, v_k$ at p. By proposition 6.1 we know that for every $1 \leq i \leq k$ it is true that $g(b_i, b_i) \colon \pi[\mathrm{Dom}(\alpha)] \to \mathbb{R}$ is a smooth function on $\pi[\mathrm{Dom}(\alpha)]$. By continuity, there exists a neighbourhood $U_1 \subseteq \pi[\mathrm{Dom}(\alpha)]$ of p on which it holds that

$$\epsilon_1 g(b_1, b_1) > 0. \tag{7.7}$$

We can then define the normalized local section on U_1

$$e_1 \colon U_1 \to E, \quad q \mapsto \frac{b_1(q)}{\sqrt{\epsilon_1 g(b_1, b_1)_p}}. \tag{7.8}$$

Also on U_1, the local section $b_2 - \epsilon_1 g(e_1, b_2) e_1$ is defined. It agrees with v_2 at p since $g_p(v_1, v_2) = 0$. Then, however, from $g_p(v_2, v_2) = \epsilon_2$ it follows that there exists a neighbourhood $U_2 \subseteq U_1$ of p such that

$$\epsilon_2\, g\left(b_2 - g\left(e_1, b_2\right) e_1, b_2 - g\left(e_1, b_2\right) e_1\right) > 0. \tag{7.9}$$

This enables us to define the normalized local section on U_2 given by

$$e_2 \colon U_2 \to E, \quad q \mapsto \frac{b_2(q) - \epsilon_1\, g\left(e_1, b_2\right)_q e_1(q)}{\sqrt{\epsilon_2\, g\left(b_2 - \epsilon_1\, g\left(e_1, b_2\right) e_1, b_2 - \epsilon_1\, g\left(e_1, b_2\right) e_1\right)_q}}. \tag{7.10}$$

In this manner, we can continue and define for $1 \leq n \leq k$, the local section

$$e_n \colon U_n \to E, \quad q \mapsto \frac{b_n(q) - \sum\limits_{m=1}^{n} \epsilon_m\, g\left(e_m, b_n\right)_q e_m(q)}{\sqrt{\epsilon_n\, g\left(b_n - \sum\limits_{m=1}^{n} \epsilon_m\, g\left(e_m, b_n\right) e_m, b_n - \sum\limits_{m=1}^{n} \epsilon_m\, g\left(e_m, b_n\right) e_m\right)_q}}. \tag{7.11}$$

In this way we obtain a finite sequence of neighbourhoods $U_k \subseteq U_{k-1} \subseteq \cdots \subseteq U_2 \subseteq U_1$ of p. It is quickly checked that, by construction, $e_1|_{U_k}, \ldots, e_k|_{U_k}$ is a local orthonormal frame.[5]

Corollary 7.2. *Let $\pi \colon E \to M$ be a vector bundle of rank k with bundle metric g of signature $(r, k - r)$.*

Due to proposition 7.1 we can reduce the structure group of the frame bundle $\hat{\pi} \colon \mathrm{Fr}(E) \to M$ to the pseudo-orthogonal group $\mathrm{O}(r, k - r)$. The principal bundle obtained in such a way is called the ***orthogonal frame bundle*** *$\hat{\pi} \colon \mathrm{Fr}_g(E) \to M$ of $\pi \colon E \to M$ with respect to g.*

Remark 7.3. As we shall see, the orthogonal frame bundle $\hat{\pi} \colon \mathrm{Fr}_g(E) \to M$ might fail to be trivial even if the frame bundle $\hat{\pi} \colon \mathrm{Fr}(E) \to M$ is trivial.

Definition 7.4. Let $\pi \colon E \to M$ be a vector bundle of rank k with a Lorentzian bundle metric g.

We say that $\pi \colon E \to M$ is **time-orientable** with respect to g if there exists a global section $X \colon M \to E$ such that $g(X_p, X_p) > 0$ for all points $p \in M$.

We will now provide an example of a three-dimensional time-orientable parallelizable spacetime with Lorentzian metric which does not admit a global orthonormal frame.

[5] Following largely the exposition found in *Finding local orthonormal frame on a Pseudo-Riemannian Manifold* by K. Lois on http://math.stackexchange.com

Example 7.1. We will depart from the three-dimensional smooth manifold $\mathbb{R}^3$. Note that $\mathbb{R}^3$ is parallelizable. Any open set of $\mathbb{R}^3$, and in particular $\mathbb{R}^3 \setminus \{0\}$, is therefore parallelizable as well.

We pick $\mathbb{R}^3 \setminus \{0\}$ as the underlying smooth manifold for our spacetime. Using spherical coordinates ($x^1 = r, x^2 = \theta, x^3 = \phi$), we equip $\mathbb{R}^3 \setminus \{0\}$ with the following Lorentzian metric:

$$g = \frac{1}{r^2}\,\mathrm{d}r \otimes \mathrm{d}r - \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\ \mathrm{d}\phi \otimes \mathrm{d}\phi\right) \tag{7.12}$$

Note that we can cover $\mathbb{R}^3 \setminus \{0\}$ using spherical coordinates corresponding to different northpole-southpole-axis and that the provided metric g is given by the above formula consistently throughout the different spherical coordinate charts. Also note that $\frac{\partial}{\partial r}$ coincides for all the possible different choices of spherical coordinates.

We observe that this spacetime is time-orientable, for we can provide a global time-like vector field given by $\frac{\partial}{\partial r}$:

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = \frac{1}{r^2} > 0. \tag{7.13}$$

The question that we would like to ask is whether there exists a **global orthonormal frame** for the metric g, remembering that $(\mathbb{R}^3 \setminus \{0\}, g)$ is both parallelizable and time-orientable?

The answer is: **No!**

This is ultimately due to the possibility that the **quotient bundle** of two trivial vector bundles might be non-trivial. In our particular case, consider the **orthogonal complement bundle** of the line-bundle $\left\{\mathbb{R}\frac{\partial}{\partial r}\right\}$ in the tangent bundle $T\left(\mathbb{R}^3 \setminus \{0\}\right)$ with respect to the metric g.

Now consider the unit-sphere S^2 as a subset of $\mathbb{R}^3 \setminus \{0\}$. For any event $p \in S^2$, the orthogonal complement of $\frac{\partial}{\partial r}$ wrt. g coincides precisely with the tangent space of S^2 at p. Hence the restriction of the orthogonal complement bundle to the subset S^2 is isomorphic to the tangent bundle of S^2.

However, it is well-known that S^2 is not parallelizable. There does not exist a global frame on S^2. Since there exists a vector bundle isomorphism between the tangent bundle TS^2 and the restriction to S^2 of the orthogonal complement bundle of the line-bundle $\left\{\mathbb{R}\frac{\partial}{\partial r}\right\}$ in the tangent bundle $T\left(\mathbb{R}^3 \setminus \{0\}\right)$, we conclude that there cannot exist a global section of the orthogonal complement bundle. In turn, there does not exist a global orthonormal frame for $(\mathbb{R}^3 \setminus \{0\}, g)$.

In this argument we cheated a little by choosing the line-bundle $\{\mathbb{R}\frac{\partial}{\partial r}\}$ from the start. It is left as an exercise to the reader to go through all the details of the construction for the case of an arbitrary global time-like vector field.

Refer to figure 7.1 for a visualization of the idea.

Subproof (g solves the Einstein equations). The question that remains is whether the provided metric g solves the Einstein equations.

Once more, we will make use of spherical coordinates ($x^1 = r, x^2 = \theta, x^3 = \phi$). The calculations, however, will be carried out using the local orthonormal frame given by:

$$h_r = r\frac{\partial}{\partial r}, \quad h_\theta = \frac{\partial}{\partial \theta}, \quad h_\phi = \frac{1}{\sin(\theta)}\frac{\partial}{\partial \phi}. \tag{7.14}$$

We will first calculate the coefficients of anholonomy according to the formula:

$$[h_a, h_b] = \Xi_{(h)}{}^{c}{}_{ab}\, h_c. \tag{7.15}$$

We obtain:

$$[h_r, h_\theta] = 0, \quad [h_r, h_\phi] = 0, \quad [h_\theta, h_\phi] = -\cot(\theta)\, h_\phi. \tag{7.16}$$

The only non-vanishing pair of coefficients of anholonomy is given by

$$\Xi_{(h)}{}^{\phi}{}_{\theta\phi} = -\Xi_{(h)}{}^{\phi}{}_{\phi\theta} = -\cot(\theta). \tag{7.17}$$

We can then calculate the coefficient functions of the *Levi-Civita* covariant derivative operator with respect to this local orthonormal frame according to the formula (use (10.27))

$$\overset{\circ}{\Gamma}_{(h)}{}^{a}{}_{bc} = \frac{1}{2}\left(\Xi_{(h)b}{}^{a}{}_{c} + \Xi_{(h)c}{}^{a}{}_{b} - \Xi_{(h)}{}^{c}{}_{ab}\right). \tag{7.18}$$

The only non-vanishing coefficients are given by:

$$-\overset{\circ}{\Gamma}_{(h)}{}^{\theta}{}_{\phi\phi} = \overset{\circ}{\Gamma}_{(h)}{}^{\phi}{}_{\theta\phi} = \cot(\theta). \tag{7.19}$$

The components of the Ricci-tensor can be calculated according to (use (10.24)):

$$\overset{\circ}{R}_{(h)ab} = h_c\left(\overset{\circ}{\Gamma}_{(h)}{}^{c}{}_{ab}\right) - h_b\left(\overset{\circ}{\Gamma}_{(h)}{}^{c}{}_{ac}\right) + \overset{\circ}{\Gamma}_{(h)}{}^{c}{}_{dc}\overset{\circ}{\Gamma}_{(h)}{}^{d}{}_{ab} - \overset{\circ}{\Gamma}_{(h)}{}^{c}{}_{db}\overset{\circ}{\Gamma}_{(h)}{}^{d}{}_{ac} - \Xi_{(h)}{}^{d}{}_{cb}\overset{\circ}{\Gamma}_{(h)}{}^{c}{}_{ad}. \tag{7.20}$$

The only non-vanishing components of the Ricci-tensor turn out to be

$$\overset{\circ}{R}_{(h)\theta\theta} = 1, \quad \overset{\circ}{R}_{(h)\phi\phi} = 1. \tag{7.21}$$

The Ricci-scalar is then found to be equal to

$$\overset{\circ}{R} = \eta^{ab}\overset{\circ}{R}_{(h)ab} = -2. \tag{7.22}$$

Finally, the Einstein-tensor can computed according to

$$\overset{\circ}{G}_{(h)ab} = \overset{\circ}{R}_{(h)ab} - \frac{R}{2}g_{(h)ab}. \tag{7.23}$$

Its only non-zero component is found to be

$$\overset{\circ}{G}_{(h)rr} = 1. \tag{7.24}$$

An observer at a point $p \in \mathbb{R}^3 \setminus \{0\}$ and with orthonormal frame (h_r, h_θ, h_ϕ) thus will measure an energy density equal to 1 but vanishing linear momentum densities and a vanishing flux of linear momentum. The matter configuration that gives rise to the spacetime in question is consequently given by non-interacting matter at rest with these observers. Note that the total matter content of this universe does not change in time. The matter is needed in order to balance the curvature of the space, resulting in a static spacetime.

Note that h_r is a no-where vanishing time-like **Killing vector field**, i.e.

$$\forall a, b \in \{r, \theta, \phi\}\colon \quad (\nabla_a h_r)_b + (\nabla_b h_r)_a = 0. \tag{7.25}$$

Remark 7.4. The counterexample from example 7.1 does not generalize to the four-dimensional case. This is due to the fact that S^3 is parallelizable, being the underlying manifold of the Lie group SU(2).

A prominent counterexample in four dimensions is provided by the extended Schwarzschild spacetime (regions I (exterior) and II (interior)) or alternatively even the maximally extended Schwarzschild spacetime (regions I, II, III and IV). The topology of both solutions is given by $S^2 \times \mathbb{R}^2$. As such, their spacetimes are parallelizable. Given the Schwarzschild metric in Kruskal coordinates, we observe that it is time-orientable. There, however, does not exist a global orthonormal frame. As we will discuss later, this has considerable consequences: Neither a spin structure nor a curvature-free metric-compatible covariant derivative operator can exist on the extended Schwarzschild spacetime.

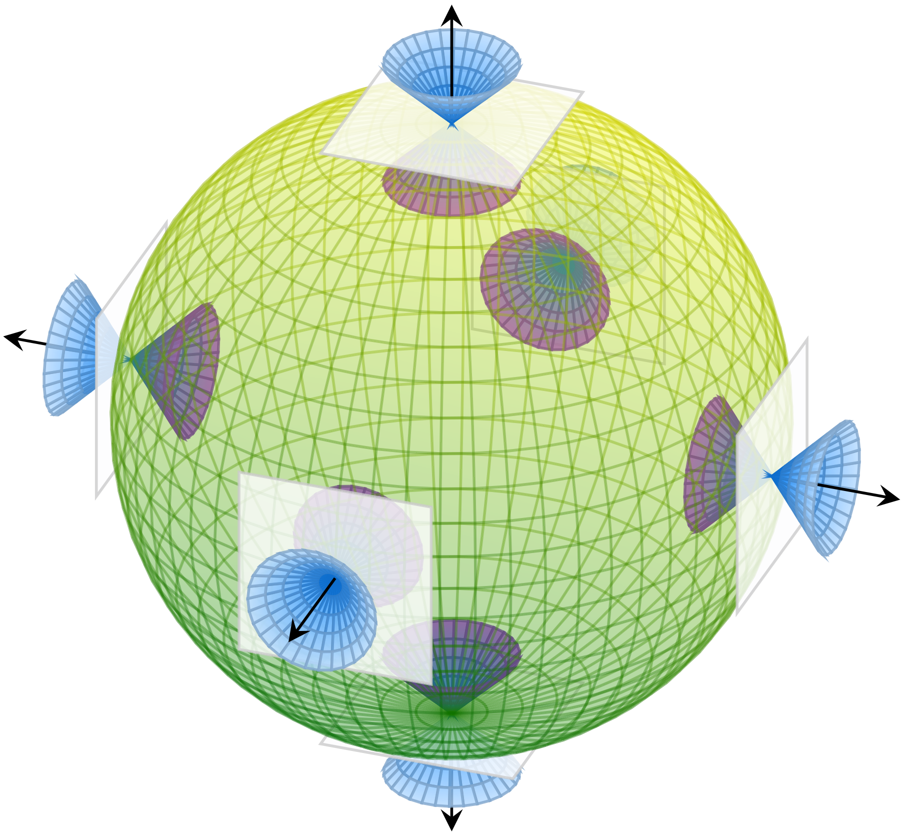

Figure 7.1: $\mathbb{R}^3 \setminus \{0\}$ with metric $g = \frac{1}{r^2}\mathrm{d}r \otimes \mathrm{d}r - (\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin^2(\theta)\ \mathrm{d}\phi \otimes \mathrm{d}\phi)$ is a time-orientable parallelizable three-dimensional spacetime without a global orthonormal frame. See example 7.1.

8 Covariant derivatives and parallel transport

Definition 8.1. Let $\pi\colon E \to M$ be a vector bundle.

A **covariant derivative operator on** $\pi\colon E \to M$ is a map

$$\nabla\colon \Gamma(TM) \times \Gamma(E) \to \Gamma(E), \quad (X, s) \mapsto \nabla_X(s) \tag{8.1}$$

that assigns to every pair of a vector field $X \in \Gamma(TM)$ on the base space and a section $s \in \Gamma(E)$ another section $\nabla_X s \in \Gamma(E)$ in such a way that for every smooth function $\varphi \in C^\infty(M)$, every two vector fields $X, X' \in \Gamma(TM)$ and every two sections $s, s' \in \Gamma(E)$ the following conditions are met:

1. Additivity with respect to the addition on $\Gamma(TM)$:

$$\nabla_{X \underset{\Gamma(TM)}{+} X'}(s) = \nabla_X(s) \underset{\Gamma(E)}{+} \nabla_{X'}(s). \tag{8.2}$$

2. $C^\infty(M)$-homogeneity with respect to the $C^\infty(M)$-multiplication on $\Gamma(TM)$:

$$\nabla_{\varphi \underset{C^\infty(M)}{\cdot} X}(s) = \varphi \underset{C^\infty(M)}{\cdot} \nabla_X(s). \tag{8.3}$$

3. Additivity with respect to the addition on $\Gamma(E)$:

$$\nabla_X \left(s \underset{\Gamma(E)}{+} s' \right) = \nabla_X(s) \underset{\Gamma(E)}{+} \nabla_X(s'). \tag{8.4}$$

4. Leibniz rule with respect to the $C^\infty(M)$-multiplication on $\Gamma(E)$:

$$\nabla_X \left(\varphi \underset{\Gamma(E)}{\cdot} s \right) = X(\varphi) \underset{C^\infty(M)}{\cdot} s \underset{\Gamma(E)}{+} \varphi \underset{C^\infty(M)}{\cdot} \nabla_X(s). \tag{8.5}$$

Note that due to item 2, ∇ is a point-operator with respect to its first argument, thus making it possible to define for any point $p \in M$ the map

$$\nabla_p\colon T_pM \times \Gamma(E) \to E_p, \quad (v, s) \mapsto (\nabla_X s)(p) \text{ where } X \in \Gamma(TM) \text{ such that } v = X_p. \tag{8.6}$$

We commonly overload notation and use the same symbol ∇ when referring to ∇_p, it is usually clear from context which map is meant.

Proposition 8.1. *Let $\pi\colon E \to M$ be a vector bundle. Given two covariant derivative operators ∇ and $\mathbb{\nabla}$ on $\pi\colon E \to M$ it holds that*

$$\nabla_X s - \mathbb{\nabla}_X s = K(X, s), \tag{8.7}$$

where $K \in \Gamma(TM^* \otimes E^{(1,1)})$ is a tensor field, a section of the tensor product bundle $TM^* \otimes E^{(1,1)}$ of the dual bundle of the tangent bundle $\pi_{TM}\colon TM \to M$ and the $(1,1)$-tensor product bundle over $\pi\colon E \to M$.

Proof 8.1. All we have to do is prove that $K(X, s)$ is $C^\infty(M)$-linear in the arguments $X \in \Gamma(TM)$ and $s \in \Gamma(E)$.

Indeed, for any two sections $X_1, X_2 \in \Gamma(TM)$ of the tangent bundle $\pi_{TM} : TM \to M$, any sections two $s_1, s_2 \in \Gamma(E)$ of $\pi : E \to M$ and any smooth function $\varphi \in C^\infty(M)$, we have

$$K(X_1 + X_2, s) = \nabla_{X_1+X_2} s - \mathbb{V}_{X_1+X_2} s, \tag{8.8}$$
$$= (\nabla_{X_1} s + \nabla_{X_2} s) - (\mathbb{V}_{X_1} s + \mathbb{V}_{X_2} s), \tag{8.9}$$
$$= K(X_1, s) + K(X_2, s), \tag{8.10}$$

$$K(X, s_1 + s_2) = \nabla_X (s_1 + s_2) - \mathbb{V}_X (s_1 + s_2), \tag{8.11}$$
$$= (\nabla_X s_1 + \nabla_X s_2) - (\mathbb{V}_X s_1 + \mathbb{V}_X s_2), \tag{8.12}$$
$$= K(X, s_1) + K(X, s_2), \tag{8.13}$$

$$K(\varphi \cdot X, s) = \nabla_{\varphi \cdot X} s - \mathbb{V}_{\varphi \cdot X} s, \tag{8.14}$$
$$= \varphi \cdot \nabla_X s - \varphi \cdot \mathbb{V}_X s, \tag{8.15}$$
$$= \varphi \cdot K(X, s), \tag{8.16}$$

$$K(X, \varphi \cdot s) = \nabla_X (\varphi \cdot s) - \mathbb{V}_X (\varphi \cdot s), \tag{8.17}$$
$$= X(\varphi) \cdot s + \varphi \cdot \nabla_X s - X(\varphi) \cdot s - \varphi \cdot \nabla_X s, \tag{8.18}$$
$$= \varphi \cdot K(X, s). \tag{8.19}$$

Remark 8.1. An immediate consequence is that the collection of covariant derivative operators on $\pi : E \to M$ forms an affine space over the unital $C^\infty(M)$-module of tensor fields $\Gamma(TM^* \otimes E^{(1,1)})$, once we allow affine spaces be modelled on unital modules over commutative unital rings.

It is due to this fact that covariant derivative operators are also historically referred to as *affine connections.*

Definition 8.2. Let $\pi : E \to M$ be a vector bundle over M of rank k that is equipped with a covariant derivative operator ∇.

Given an open subset $U \in \mathcal{O}_M$ of M, a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)} \in \Gamma(TU)$ for TM over U and a local frame $b_1, \ldots, b_k \in \Gamma(E|_U)$ for E over U we define the **coefficient functions of the covariant derivative operator ∇ with respect to the local frame $e_1, \ldots, e_k$ for TM over U and the local frame $b_1, \ldots, b_k$ for E over U** by

$$\Gamma_{(e,b)}{}^m{}_{nj} = b^m \left(\nabla_{e_j} b_n\right), \tag{8.20}$$

where $b^1, \ldots, b^k \in \Gamma(E^*|_U)$ denotes the coframe of $b_1, \ldots, b_k$.

Exercise 8.1. For every vector field $X \in \Gamma(TM)$ and every section $s \in \Gamma(E)$ it holds on U that

$$\nabla_X s = \left(X(s^m_{(b)}) + \Gamma_{(e,b)}{}^m{}_{nj} X^j_{(e)} s^n_{(b)}\right) b_m. \tag{8.21}$$

Exercise 8.2.

Let $p \in M$ and let $U \in \mathcal{O}_M$ be a neighbourhood of p together with a local frame $b_1, \ldots, b_k$ of $\pi : E \to M$ over U and a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ of $\pi_{TM} : TM \to M$ over U.

Let $V \in \mathcal{O}_M$ be another neighbourhood of p together with another local frame $\tilde{b}_1, \ldots, \tilde{b}_k$ of $\pi : E \to M$ over V and another local frame $\tilde{e}_1, \ldots, \tilde{e}_{\mathrm{Dim}(M)}$ of $\pi_{TM} : TM \to M$ over V. Denote by $B : U \cap V \to \mathrm{GL}(k, \mathbb{R})$ and $E : U \cap V \to \mathrm{GL}(\mathrm{Dim}(M), \mathbb{R})$ the smooth maps such that $\tilde{b}_1 = B^m{}_1\, b_m, \ldots, \tilde{b}_k = B^m{}_k\, b_m$ and $\tilde{e}_1 = E^m{}_1\, e_m, \ldots, \tilde{e}_{\mathrm{Dim}(M)} = E^m{}_{\mathrm{Dim}(M)}\, e_m$ on $U \cap V$.

Show that the coefficient functions of the covariant derivative operator ∇ with respect to (e, b) and $(\tilde{e}, \tilde{b})$, respectively, satisfy the relation

$$\Gamma_{(\tilde{e},\tilde{b})}{}^{m}{}_{ni} = B_k{}^m \, E^j{}_i \left(B^l{}_n \, \Gamma_{(e,b)}{}^{k}{}_{lj} + e_j \left(B^k{}_n \right) \right), \tag{8.22}$$

where $B_k{}^m = (B^{-1})^m{}_k$ denotes the inverse of B.

Proposition 8.2. *Let ∇ be a covariant derivative operator on a vector bundle $\pi\colon E \to M$ of rank k.*

Given a smooth map $f\colon N \to M$, the covariant derivative operator ∇ on $\pi\colon E \to M$ canonically induces a covariant derivative operator ∇^f on the pullback bundle f^*E.

Proof 8.2. Recall the equivalence of sections along f and sections of the pullback bundle. In order to avoid heavy notation, we will work with sections along f throughout this proof.

For any section $s'\colon M \to E$ and every vector $X_p \in TN$ at a point $p \in N$, we require

$$\nabla^f_{X_p}(s' \circ f) \coloneqq \nabla_{f_* X_p} s'. \tag{8.23}$$

We then extend the definition to any section $s\colon N \to E$ along f. To this end, suppose $p \in N$ and let $U \in \mathcal{O}_M$ be a neighbourhood of $f(p)$ together with a local frame $b_1, \ldots, b_k$ of $\pi\colon E \to M$ over U and a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ of $\pi_{TM}\colon TM \to M$ over U. Then, we extend using the Leibniz rule

$$\begin{aligned} \nabla^f_{X_p} s &= \nabla^f_{X_p}(s^m (b_m \circ f)), \\ &\coloneqq X_p(s^m) + s^m(p) \, \nabla^f_{X_p}(b_m \circ f), \\ &= X_p(s^m) + s^m(p) \, \nabla_{f_* X_p} b_m. \end{aligned} \tag{8.24}$$

If we can show that ∇^f can be expressed consistently using coefficient functions, satisfying the corresponding translation formula, then the items 1 to 4 from definition 8.1 are readily verified.

To this end, let $p \in N$ and $W \in \mathcal{O}_N$ be a neighbourhood of p such that $f[W] \subseteq U$ together with a local frame $a_1, \ldots, a_{\mathrm{Dim}(N)}$ of $\pi_{TN}\colon TN \to N$ over W. Then, a short calculation yields

$$\nabla^f_{a_{i,p}}(b_n \circ f) = (f_{*,p} a_{i,p})^l \left(\Gamma_{(e,b)}{}^{m}{}_{nl} \circ f \right)(p) \; b_n(f(p)). \tag{8.25}$$

So the coefficient functions of ∇^f satisfy

$$\Gamma^f_{(a,b\circ f)}{}^{m}{}_{ni} = (f_* \circ a_i)^l \left(\Gamma_{(e,b)}{}^{m}{}_{nl} \circ f \right), \tag{8.26}$$

and consequently, are smooth.

Now suppose that we are given other neighbourhoods $\tilde{W} \in \mathcal{O}_N$ of p and $\tilde{U} \in \mathcal{O}_M$ of $f(p)$ satisfying $f[\tilde{W}] \subseteq \tilde{U}$ together with a local frame $\tilde{b}_1, \ldots, \tilde{b}_k$ of $\pi\colon E \to M$ on $\tilde{U}$ and another local frame $\tilde{a}_1, \ldots, \tilde{a}_{\mathrm{Dim}(N)}$ of $\pi_{TN}\colon TN \to N$ on $\tilde{W}$. There exist smooth maps $B\colon U \cap \tilde{U} \to \mathrm{GL}(k, \mathbb{R})$ and $A\colon W \cap \tilde{W} \to \mathrm{GL}(\mathrm{Dim}(N), \mathbb{R})$ such that $\tilde{b}_1 = B^m{}_1 \, b_m, \ldots, \tilde{b}_k = B^m{}_k \, b_m$ and $\tilde{a}_1 = A^m{}_1 \, a_m, \ldots, \tilde{a}_{\mathrm{Dim}(N)} = A^m{}_{\mathrm{Dim}(N)} \, a_m$ on $W \cap \tilde{W}$. The coefficient functions satisfy:

$$\begin{aligned} \Gamma^f_{(\tilde{a},\tilde{b}\circ f)}{}^{m}{}_{ni} &= (f_* \circ \tilde{a}_i)^l \left(\Gamma_{(e,\tilde{b})}{}^{m}{}_{nl} \circ f \right), \\ &= \left(f_* \circ (A^j{}_i a_j) \right)^l (B_r{}^m \circ f) \left((B^s{}_n \circ f) \left(\Gamma_{(e,b)}{}^{r}{}_{sl} \circ f \right) + \left(e_l \left(B^r{}_n \right) \circ f \right) \right), \\ &= (f_* \circ a_j)^l \, A^j{}_i \, (B_r{}^m \circ f) \left((B^s{}_n \circ f) \left(\Gamma_{(e,b)}{}^{r}{}_{sl} \circ f \right) + \left(e_l \left(B^r{}_n \right) \circ f \right) \right), \\ &= A^j{}_i (B_r{}^m \circ f) \left((B^s{}_n \circ f) \, \Gamma^f_{(a,b\circ f)}{}^{r}{}_{sj} + (f_* \circ a_i) \left(B^r{}_n \right) \right), \\ &= A^j{}_i (B_r{}^m \circ f) \left((B^s{}_n \circ f) \, \Gamma^f_{(a,b\circ f)}{}^{r}{}_{sj} + a_i \left(B^r{}_n \circ f \right) \right). \end{aligned} \tag{8.27}$$

This is the expected transformation formula. This concludes the proof.

Definition 8.3. Let $p, q \in M$ be points in M.

A path $\gamma\colon [0,1] \to M$ is said to be **piecewise smooth** if there exist finitely many numbers $t_0 = 0 < t_1 < \cdots < t_n = 1$ such that for each $1 \le i \le n$ the restriction $\gamma_{[t_{i-1},t_i]}\colon [t_{i-1}, t_i] \to M$ is smooth.

Definition 8.4. Let $\gamma_1\colon [0,1] \to M$ and $\gamma_2\colon [0,1] \to M$ be two piecewise smooth paths in a smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ such that $\gamma_1(1) = \gamma_2(0)$ and $\xi \in (0,1)$.

The **ξ-concatenation** of γ_1 with γ_2 is the piecewise smooth path

$$\gamma_1 \overset{\xi}{*} \gamma_2\colon [0,1] \to M, \quad t \mapsto \begin{cases} \gamma_1\left(\frac{t}{\xi}\right) & \text{for } t \le \xi \\ \gamma_2\left(\frac{t-\xi}{1-\xi}\right) & \text{for } t > \xi \end{cases}. \tag{8.28}$$

The $\frac{1}{2}$-concatenation of γ_1 with γ_2 will usually be abbreviated using the short-hand notation $\gamma_1 * \gamma_2$.

Definition 8.5. Let $\gamma\colon [0,1] \to M$ be a piecewise smooth path in a smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$.

The **path inversion** of γ is the piecewise smooth path

$$\overline{\gamma}\colon [0,1] \to M, \quad t \mapsto \gamma(1-t). \tag{8.29}$$

Definition 8.6. A smooth path $\gamma\colon [0,1] \to M$ is said to be **regular** if for every $t \in [0,1]$ its velocity $\dot\gamma(t)$ is different from zero, i.e., if $\dot\gamma(0) \neq 0$.

Definition 8.7. Let $\pi\colon E \to M$ be a vector bundle over M of rank k.

A **parallel transport system** $\mathbb{P}$ on $\pi\colon E \to M$ is a map

$$\mathbb{P}\colon \{(\gamma, a) \in C^\infty([0,1], M) \times E \mid \gamma(0) = \pi(a)\} \to E, \quad (\gamma, a) \mapsto \mathbb{P}_\gamma(a) \tag{8.30}$$

that assigns to every smooth path $\gamma\colon [0,1] \to M$ and every vector $a \in E_{\gamma(0)}$ in the fibre over $\gamma(0)$ a vector $\mathbb{P}_\gamma(a) \in E_{\gamma(1)}$ in the fibre over $\gamma(1)$ such that:

1. For every smooth path $\gamma\colon [0,1] \to M$ the map $\mathbb{P}_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ is a linear isomorphism. (See fig. 8.1.)
2. For every smooth path $\gamma\colon [0,1] \to M$ the parallel transport $\mathbb{P}_{\overline{\gamma}}$ along its path inversion coincides with $\mathbb{P}_\gamma^{-1}$.
3. For every two smooth paths γ_1, γ_2 satisfying $\gamma_1(1) = \gamma_2(0)$, the parallel transport $\mathbb{P}_{\gamma_1 \overset{\xi}{*} \gamma_2}$ along the ξ-concatenation $\gamma_1 \overset{\xi}{*} \gamma_2$ coincides with $\mathbb{P}_{\gamma_2} \circ \mathbb{P}_{\gamma_1}$.
4. For every smooth path $\gamma\colon [0,1] \to M$ and every reparametrization $\varphi\colon \mathrm{Diff}([0,1])$ satisfying $\varphi(0) = 0$ and $\varphi(1) = 1$, the parallel transport $\mathbb{P}_{\gamma\circ\varphi}$ along the reparametrized path $\gamma \circ \varphi$ coincides with $\mathbb{P}_\gamma$.
5. For every open set $U \in \mathcal{O}_M$ and every smooth map $\Psi\colon TU \to U$ satisfying $\Psi(0_{T_pM}) = p$ for all $p \in U$, the map
$$TU \oplus E|_U \to E, \quad (X, a) \mapsto \mathbb{P}_{t\mapsto\Psi(tX)}(a) \tag{8.31}$$
is smooth.
6. For every smooth path $\gamma\colon [0,1] \to M$, the map
$$P_{\gamma,a}\colon [0,1] \to E, \quad t \mapsto \mathbb{P}_{s\mapsto\gamma(st)}(a) \tag{8.32}$$
is smooth.
7. For every two smooth paths γ_1, γ_2 leaving the point $\gamma_1(0) = \gamma_2(0)$ with the same velocity $\dot\gamma_1(0) = \dot\gamma_2(0)$, the parallel transports along γ_1 and γ_2 satisfy for every $a \in E_{\gamma_1(0)}$ the relation
$$\dot P_{\gamma_1,a}(0) = \dot P_{\gamma_2,a}(0). \tag{8.33}$$

Remark 8.2. As we will see shortly, a weaker version - only valid for regular or constant paths - of item 6 from definition 8.7 already follows from item 5. This is the extent of proposition 8.5 below. The reason we do require item 6 is that we would like to avoid having to exclude smooth paths that are neither regular nor constant every time we talk about parallel transport systems.

In fact, we could omit item 6 due to the fact that the proofs of equivalence between covariant derivative operators and parallel transport systems - theorems 8.8 and 8.9 - do not rely on item 6. Item 6 would then follow from the proof of theorem 8.8. In order not to obscure the true nature of what a parallel transport system encompasses, we choose to include item 6 explicitly.

Lemma 8.3. *Suppose $\gamma\colon [0,1] \to M, t \mapsto p$ for $p \in M$ is a constant path.*

Then the parallel transport along γ is trivial, i.e,

$$\mathbb{P}_\gamma = \mathrm{id}_{E_p} . \tag{8.34}$$

Proof 8.3. First, observe that the path concatenation $\gamma * \gamma$ coincides with γ. As a consequence, item 3 from definition 8.7 infers that

$$\mathbb{P}_\gamma = \mathbb{P}_{\gamma*\gamma} = \mathbb{P}_\gamma \circ \mathbb{P}_\gamma. \tag{8.35}$$

However, since γ coincides with its path inversion $\overline{\gamma}$, item 2 from definition 8.7 states that

$$\mathbb{P}_\gamma = \mathbb{P}_{\overline{\gamma}} = \mathbb{P}_\gamma^{-1}. \tag{8.36}$$

Substituted in the right hand side of equation (8.35), we obtain

$$\mathbb{P}_\gamma = \mathrm{id}_{E_p}, \tag{8.37}$$

as claimed.

Lemma 8.4. *Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ be a smooth manifold, $p \in M$ a point and $x \in \mathcal{A}_M$ a chart with the property that $x[\mathrm{Dom}(x)] = \mathbb{R}^{\mathrm{Dim}(M)}$. Note that this can always be achieved. (We can start by removing the offset of x at p in order to achieve $x(p) = 0$. Next, we can restrict the codomain $x[\mathrm{Dom}(x)]$ to a star-shaped open set, which happens to be diffeomorphic to $\mathbb{R}^{\mathrm{Dim}(M)}$.)*

The map

$$\Psi^{(x)}\colon \mathrm{T}\,\mathrm{Dom}(x) \to \mathrm{Dom}(x), \quad X \mapsto x^{-1}\left(x(\pi_{TM}(X)) + x_*(X)\right) \tag{8.38}$$

is smooth and satisfies $\Psi^{(x)}(0_{T_qM}) = q$ for all $q \in \mathrm{Dom}(x)$.

Moreover, for every $X \in T\,\mathrm{Dom}\,x$ the smooth path

$$\gamma_X^{(x)}\colon [0,1] \to \mathrm{Dom}(x), \quad t \mapsto \Psi^{(x)}(tX) \tag{8.39}$$

satisfies $\dot{\gamma}_X^{(x)}(0) = X$.

Proof 8.4. First of all, $\Psi^{(x)}$ is smooth since $x_*\colon T\,\mathrm{Dom}(x) \to \mathbb{R}^{\mathrm{Dim}(M)}$ is a vector bundle chart for the tangent bundle $\pi_{TM}\colon TM \to M$ and $x\colon \mathrm{Dom}(x) \to x[\mathrm{Dom}(x)]$ is a diffeomorphism.

For the first part it suffices to plug in $x_*(0_{T_qM}) = 0$, in order to obtain

$$\Psi^{(x)}(q) = x^{-1}(x(\pi_{TM}(0_{T_qM}))) = q. \tag{8.40}$$

For the second part, observe that, by definition of $\Psi^{(x)}$, it holds that

$$\left(x \circ \gamma_X^{(x)}\right)(t) = x(\pi_{TM}(X)) + x_*(tX) = x(\pi_{TM}(X)) + t \cdot x_*(X). \tag{8.41}$$

But then, the velocity $\dot{\gamma}_X^{(x)}(0)$ satisfies

$$x_*\left(\dot{\gamma}_X^{(x)}(0)\right) = \left(x \circ \gamma_X^{(x)}\right)'(0) = x_*(X). \tag{8.42}$$

Finally, since $\gamma_X^{(x)}(0) = \pi_{TM}(X)$, it follows that $\dot{\gamma}_X^{(x)}(0) = X$.

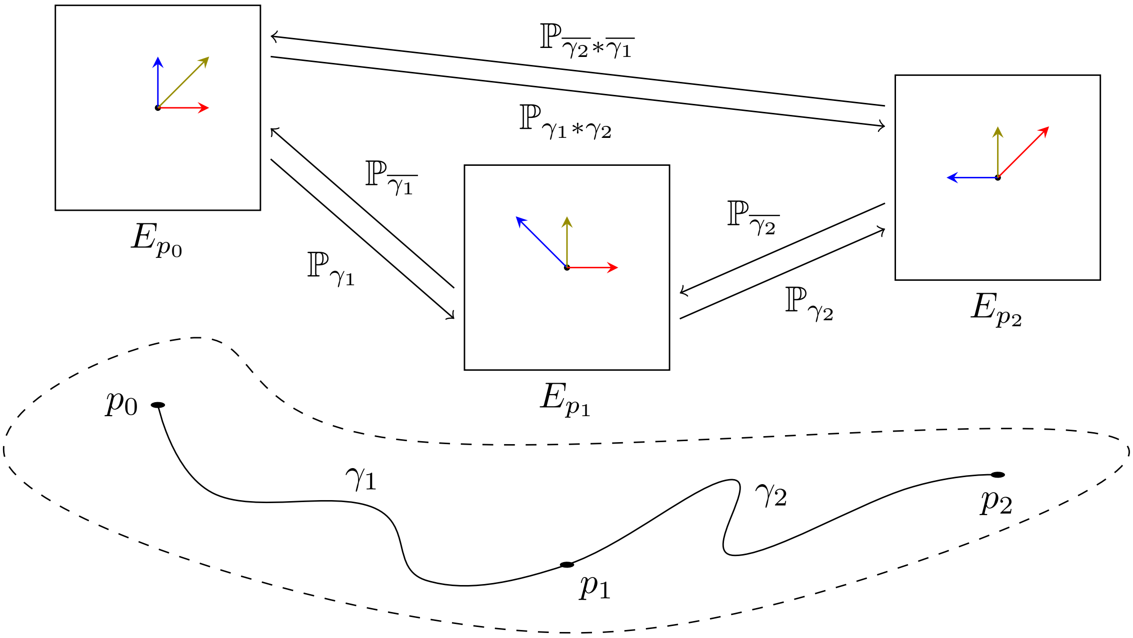

Figure 8.1: Illustriation of the meaning of items 1 to 3 from the definition of a parallel transport system.

Proposition 8.5. *From item 5 from definition 8.7 it follows that for every regular or constant path $\gamma\colon [0,1] \to M$ and every point $a \in E_{\gamma(0)}$, the path*

$$P_{\gamma,a}\colon [0,1] \to E, \quad t \mapsto \mathbb{P}_{s\mapsto\gamma(ts)}(a) \tag{8.43}$$

is smooth.

Proof 8.5. Let $t \in [0,1]$. In order to prove that $P_{\gamma,a}\colon [0,1] \to E$ is a smooth path in E for every $a \in E_{\gamma(0)}$, we have to distinguish between the parallel transport along a regular path and the parallel transport along a constant path.

Subproof (Case 1: $\gamma\colon [0,1] \to M$ is regular). Recall that for every point t in the $[0,1]$ the velocity $\dot\gamma(t)$ is different from zero. In fact, there exists $\epsilon > 0$ such that there exists an extension $\gamma|_{(-\epsilon,1+\epsilon)}\colon (-\epsilon, 1+\epsilon) \to M$ is an immersion. It might, however, fail to be an injection.

Let us show that for any $t \in [0,1]$, the curve $P_{\gamma,a}$ is smooth at t. There exists $\epsilon > 0$ such that $\gamma|_{(t-\epsilon,t+\epsilon)}$ is an injective immersion, and as such qualifies as an embedding. It is well-known[6] that we can find a slice chart for $\gamma|_{(t-\epsilon,t+\epsilon)}$ at $\gamma(t)$: a chart $x \in \mathcal{A}_M$ at $\gamma(t)$ such that $x^2(\gamma(s)) = \cdots = x^{\mathrm{Dim}(M)}(\gamma(s)) = 0$ for all $s \in (t-\delta, t+\delta)$, where $\epsilon > \delta > 0$. We may adapt x such that it satisfies the hypothesis of lemma 8.4 in order to use the map $\Psi^{(x)}\colon T\,\mathrm{Dom}(x) \to \mathrm{Dom}(x)$. Observe that for every $r \in (-\epsilon, \epsilon)$ it holds that

$$\Psi^{(x)}(r\dot\gamma(t)) = \gamma(t+r). \tag{8.44}$$

Then, however, for every $b \in E_{\gamma(t)}$ the map

$$(-\epsilon, \epsilon) \to E, \quad r \mapsto \mathbb{P}_{s\mapsto\Psi^{(x)}(sr\dot\gamma(t))}(b) \tag{8.45}$$

[6] *Introduction to Smooth Manifolds* (2nd edition) by J. M. Lee, Page 101, Theorem 5.8 (Local Slice Criterion for Embedded Submanifolds)

smoothly depends on r. This is an interesting result due to the fact that the path $s \mapsto \Psi^{(x)}(sr\dot{\gamma}(t))$ coincides with the path

$$\gamma^t_{+r}\colon [0,1] \to M, \quad s \mapsto \gamma(t+sr). \tag{8.46}$$

Let us also introduce the definition of the path

$$\gamma_t\colon [0,1] \to M, \quad s \mapsto \gamma(st). \tag{8.47}$$

The $\frac{t}{t+r}$-concatenation of γ_t with γ^T_{+r} is equal to the path γ_{t+r}. The parallel transport $\mathbb{P}_{\gamma_t \overset{\frac{t}{t+r}}{*} \gamma^t_{+r}}$ along the $\frac{t}{t+r}$-path concatenation $\gamma_t \overset{\frac{t}{t+r}}{*} \gamma^t_{+r}$ coincides with $\mathbb{P}_{\gamma^t_{+r}} \circ \mathbb{P}_{\gamma_t}$ due to item 3 of definition 8.7. In short, we have that $\mathbb{P}_{\gamma^t_{+r}} \circ \mathbb{P}_{\gamma_t}$ is equal to $\mathbb{P}_{\gamma_{t+r}}$. This finally proves that in the vicinity $(t-\epsilon, t+\epsilon)$ of t, the map

$$P_{\gamma,a}\colon [0,1] \to E, \quad \tau \mapsto \mathbb{P}_{\gamma_\tau}(a) \tag{8.48}$$

is smooth for any $a \in E_{\gamma(0)}$.

Subproof (Case 2: $\gamma\colon [0,1] \to M$ is smooth). Let $r \in [t-\epsilon, t+\epsilon]$. The path $\gamma^t_{+r}\colon [0,1] \to M, s \mapsto \gamma(t+sr)$ is constant. By lemma 8.3, the parallel transport along γ^t_{+r} is trivial, i.e.,

$$\mathbb{P}_{\gamma^t_{+r}} = \mathrm{id}_{E_{\gamma(t)}}\,. \tag{8.49}$$

It is immediate that for every $b \in E_{\gamma(t)}$ the constant map

$$(-\epsilon, \epsilon) \to E \quad r \mapsto \mathbb{P}_{\gamma^t_{+r}}(b) \tag{8.50}$$

is smooth. In just the same way as done for case 1 we can establish that $\mathbb{P}_{\gamma^t_{+r}}(\mathbb{P}_{\gamma_t}(a))$ coincides with $\mathbb{P}_{\gamma_{t+r}}(a)$ for any $a \in E_{\gamma(0)}$. This proves that in the vicinity $[t-\epsilon, t+\epsilon]$ of t, the map

$$P_{\gamma,a}\colon [0,1] \to E, \quad \tau \mapsto \mathbb{P}_{\gamma_\tau}(a) \tag{8.51}$$

is smooth. This concludes the proof.

Lemma 8.6. *Let $p \in M$ and $a \in E_p$. The map*

$$L_a\colon T_pM \to T_aE, \quad X \mapsto \dot{P_{\gamma_X,a}}(0), \text{ where } \gamma_X : \mathbb{R} \to M \text{ such that } X = \dot{\gamma}_X(0), \tag{8.52}$$

is a linear injection.

Proof 8.6. The map $L_a\colon T_pM \to T_aE$ is well-defined due to item 7 from definition 8.7. In order to show that L_a is linear, we will make use of lemma 8.4. Suppose $x \in \mathcal{A}_M$ is a map at p such that $x[\mathrm{Dom}(x)] = \mathbb{R}^{\mathrm{Dim}(M)}$ and consider the map $\Psi^{(x)}\colon T\,\mathrm{Dom}(x) \to \mathrm{Dom}(x)$ from lemma 8.4 that satisfies the hypothesis of item 5 from definition 8.7. For every $X \in T_pM$, define the smooth path (see lemma 8.4)

$$\gamma^{(x)}_X\colon [0,1] \to M, \quad t \mapsto \Psi^{(x)}(tX). \tag{8.53}$$

We can define the auxiliary map

$$C_a\colon T_pM \to E, \quad X \mapsto \mathbb{P}_{s\mapsto\Psi^{(x)}(sX)}(a). \tag{8.54}$$

It is smooth by item 5 from definition 8.7. In terms of C_a, we can express L_a in the following fashion:

$$\begin{aligned} L_a(X) &= \dot{P_{\gamma^{(x)}_X,a}}(0), \\ &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} \left(\mathbb{P}_{s\mapsto\Psi^{(x)}(stX)}(a)\right), \\ &= C_{a*,0_{T_pM}}\left(\mathcal{J}_{0_{T_pM}}(X)\right). \end{aligned} \tag{8.55}$$

However, $C_{a*,0_{T_pM}}\colon T_{0_{T_pM}}(T_pM) \to T_aE$ and the canonical isomorphism $\mathcal{J}_{0_{T_pM}}\colon T_pM \to T_{0_{T_pM}}(T_pM)$ are linear maps, proving that $L_a\colon T_pM \to T_aE$ is linear.

Moreover, $L_a: T_pM \to T_aE$ is injective. For, if $X \in T_pM$, then $\pi \circ P_{\gamma_X,a} = \gamma_X$ and, consequently, $(\pi_* \circ L_a)(X) = X$. This proves that $\pi_* \circ L_a = \mathrm{id}_{T_pM}$ and that $L_a: T_pM \to T_aE$ is injective.

Lemma 8.7. *Let $\pi: E \to M$ be a vector bundle of rank k. Given a point $p \in M$, a vector bundle chart $\alpha: \mathrm{Dom}(\alpha) \to \mathbb{R}^k$ at p, and points $a, Y \in E_p$ in the fibre over p, it holds that*

$$\mathrm{d}\alpha\left(\mathcal{J}_a(Y)\right) = \alpha(Y), \tag{8.56}$$

where $\mathcal{J}_a: E_p \to T_a(E_p)$ is the canonical isomorphism.

Proof 8.7. Recall the curve

$$\gamma_Y^a: \mathbb{R} \to E_p, \quad t \mapsto a \underset{E_p}{+} t \underset{E_p}{\cdot} Y \tag{8.57}$$

in E_p. It gives rise to the curve

$$\alpha \circ \gamma_Y^a: \mathbb{R} \to \mathbb{R}^k, \quad t \mapsto \alpha(a) + t \cdot \alpha(Y) \tag{8.58}$$

in $\mathbb{R}^k$. Here we used that $\alpha|_{E_p}: E_p \to \mathbb{R}^k$ is linear. Then, it follows that for every $1 \leq i \leq k$ it holds that

$$\mathrm{d}\alpha^i\left(\mathcal{J}_a(Y)\right) = \left(\alpha^i \circ \gamma_Y^a\right)'(0) = \alpha^i(Y). \tag{8.59}$$

Definition 8.8. A section $Y \in \Gamma_\gamma(E)$ along a curve $\gamma: (\tau_1, \tau_2) \to M$ is said to be **parallelly transported along** γ if for every $t_1, t_2 \in (\tau_1, \tau_2)$ it holds that

$$Y(t_2) = \mathbb{P}_{s \mapsto \gamma(t_1 + s(t_2 - t_1))}(Y(t_1)). \tag{8.60}$$

Theorem 8.8. *Let $\pi: E \to M$ be a vector bundle of rank k equipped with a covariant derivative operator ∇.*

We can canonically construct a parallel transport system $\mathbb{P}$ on $\pi: E \to M$ such that a section Y along a curve $\gamma: (t_1, t_2) \to M$ is parallelly transported along γ if and only if $\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0_{\Gamma_\gamma(E)}$.

Proof 8.8. For every smooth path $\gamma: [0,1] \to M$ and every element $a \in E_{\gamma(0)}$ in the fibre over the starting point $\gamma(0)$ denote by $P_{\gamma,a}: [0,1] \to E$ the unique section along γ with $P_{\gamma,a}(0) = a$ such that $\nabla^\gamma_{\partial_{\mathrm{id}}} P_{\gamma,a} = 0$.

For every $t \in [0,1]$ there exists a neighbourhood U of $\gamma(t)$ together with a local frame $b_1, \ldots, b_k$ of E and a local frame $e_1, \ldots, e_k$ of TM. There also exists $\epsilon > 0$ such that $\gamma[(t-\epsilon, t+\epsilon)] \subseteq U$. Using the vector bundle chart $\alpha \in \mathcal{B}$ associated to the local frame $b_1, \ldots, b_k$, we can express the equation $\nabla^\gamma_{\partial_{\mathrm{id}}} P_{\gamma,a} = 0$ on $s \in (t-\epsilon, t+\epsilon)$ as

$$\left(\alpha^m \circ P_{\gamma,a}\right)'(s) + {\Gamma^\gamma_{(\partial_{\mathrm{id}}, b \circ \gamma)}}^m{}_n(s) \cdot \left(\alpha^n \circ P_{\gamma,a}\right)(s) = 0, \tag{8.61}$$

where we made use of the coefficient functions (which are smooth!) of the covariant derivative operator ∇^γ along γ and the identity chart id of $\mathbb{R}$. The above equation is a linear homogeneous first-order differential equation with variable coefficients. It is well-known[7] that there exists a unique (smooth) solution $\alpha \circ P_{\gamma,a}: (t-\epsilon, t+\epsilon) \to \mathbb{R}^k$. Moreover, the solutions of equation (8.61) form a subspace of $C^\infty((t-\epsilon, t+\epsilon), \mathbb{R}^k)$. Since the closed interval $[0,1]$ is compact, we can cover the interval $[0,1]$ by a finite number of such intervals $(t-\epsilon, t+\epsilon)$. Gluing them together appropriately leads us to conclude that there exists a unique section $P_{\gamma,a}$ along γ such that $P_{\gamma,a}(0) = a$ and $\nabla^\gamma_{\partial_{\mathrm{id}}} P_{\gamma,a} = 0$.

Having established this, we define:

$$\mathbb{P}: \{(\gamma, a) \in C^\infty([0,1], M) \times E \mid \gamma(0) = \pi(a)\} \to E, \quad (\gamma, a) \mapsto \mathbb{P}_\gamma(a) := P_{\gamma,a}(1). \tag{8.62}$$

It is left to prove that $\mathbb{P}$ satisfies the items 1 to 5 and 7 from definition 8.7. Item 6 is satisfied by construction.

[7] *Ordinary Differential Equations* by V. I. Arnol'd, Page 241, Chapter 3, Section 27, Subsections 1-3

Subproof (4. Reparametrization invariance). Let $\varphi\colon [0,1] \to [0,1]$ be a bijective smooth map that satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. There exists $\delta > 0$ such that $\varphi[(\varphi^{-1}(t) - \delta, \varphi^{-1}(t) + \delta)] \subseteq (t - \epsilon, t + \epsilon)$. For every $r \in (\varphi^{-1}(t) - \delta, \varphi^{-1}(t) + \delta)$, the coefficient functions along $\gamma \circ \varphi$ satisfy

$$\Gamma^{\gamma\circ\varphi}_{(\partial_{\mathrm{id}}, b\circ\gamma\circ\varphi)}{}^m{}_n(r) = \left(\Gamma^{\gamma}_{(\partial_{\mathrm{id}}, b\circ\gamma)}{}^m{}_n \circ \varphi\right)(r) \cdot \varphi'(r). \tag{8.63}$$

This has the consequence that $P_{\gamma\circ\varphi,a}$ and $P_{\gamma,a} \circ \varphi$ satisfy the same differential equation for each vector bundle chart. For, if we multiply the equation for $P_{\gamma,a}$ at $\varphi(r)$

$$(\alpha^m \circ P_{\gamma,a})'(\varphi(r)) + \Gamma^{\gamma}_{(\partial_{\mathrm{id}}, b\circ\gamma)}{}^m{}_n(\varphi(r)) \cdot (\alpha^n \circ P_{\gamma,a})(\varphi(r)) = 0, \tag{8.64}$$

by $\varphi'(r)$, then, by means of equation (8.63), we obtain

$$(\alpha^m \circ P_{\gamma,a} \circ \varphi)'(r) + \Gamma^{\gamma\circ\varphi}_{(\partial_{\mathrm{id}}, b\circ\gamma\circ\varphi)}{}^m{}_n(r) \cdot (\alpha^n \circ P_{\gamma,a} \circ \varphi)(r) = 0, \tag{8.65}$$

the same equation that is satisfied by $P_{\gamma\circ\varphi,a}$. By uniqueness of the solution and since $P_{\gamma\circ\varphi,a}(0) = a = P_{\gamma,a}(0)$, we conclude that $P_{\gamma\circ\varphi,a} = P_{\gamma,a} \circ \varphi$. Then also $P_{\gamma\circ\varphi,a}(1) = P_{\gamma,a}(1)$ and consequently $\mathbb{P}_{\gamma\circ\varphi}(a) = \mathbb{P}_{\gamma}(a)$.

Subproof (2. Path inversion). A special case of equation (8.63) is obtained when inspecting the path inverting substitution $\varphi\colon \mathbb{R} \to \mathbb{R}, s \mapsto 1 - s$. Then:

$$\forall s \in (t - \epsilon, t + \epsilon)\colon \quad \Gamma^{\overline{\gamma}}_{(\partial_{\mathrm{id}}, b\circ\overline{\gamma})}{}^m{}_n(1 - s) = -\Gamma^{\gamma}_{(\partial_{\mathrm{id}}, b\circ\gamma)}{}^m{}_n(s). \tag{8.66}$$

This ensures that under the substitution $\varphi\colon \mathbb{R} \to \mathbb{R}, s \mapsto 1 - s$, the curve $P_{\gamma,a} \circ \varphi$ agrees with $P_{\overline{\gamma}, P_{\gamma,a}(1)}$ and vice versa. Then, however, it is true that $\mathbb{P}_{\overline{\gamma}} = \mathbb{P}_{\gamma}^{-1}$.

Subproof (1. Linear isomorphism). As a corollary of the above subproof, it is established that $\mathbb{P}_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ is a bijection. Furthermore, since on every interval contained in a vector bundle chart the solutions form a vector space, together with the fact that we always succeed to cover the interval $[0,1]$ by a finite number of intervals ($[0,1]$ is compact), the map $\mathbb{P}_\gamma\colon E_{\gamma(0)} \to E_{\gamma(1)}$ is in particular a linear isomorphism.

Subproof (3. Path concatenation). We are essentially making use of equation (8.63) and making contact with the conclusion of the subproof that proved reparametrization invariance.

Suppose we are given two smooth paths $\gamma_1\colon [0,1] \to M$ and $\gamma_2\colon [0,2] \to M$ such that $\gamma_1(1) = \gamma_2(0)$ and with the property that their ξ-path concatenation $\gamma_1 \between_\xi \gamma_2$ is also a smooth path. Define the smooth maps $\mu_1\colon [0,1] \to [0,1], s \mapsto \xi s$ and $\mu_2\colon [0,1] \to [0,1], s \mapsto \xi + (1 - \xi)\, s$. Observe that $(\gamma_1 \between_\xi \gamma_2) \circ \mu_1 = \gamma_1$ and $(\gamma_1 \between_\xi \gamma_2) \circ \mu_2 = \gamma_2$. Part of the conclusion from subproof 8.8.1 tells us that $P_{\gamma_1 \between_\xi \gamma_2, a} \circ \mu_1$ agrees with $P_{\gamma_1,a}$. Consequently, $P_{\gamma_1 \between_\xi \gamma_2, a}(\xi) = P_{\gamma_1,a}(1) = \mathbb{P}_\gamma(a)$. Analogously, we know that $P_{\gamma_1 \between_\xi \gamma_2, a} \circ \mu_2$ coincides with $P_{\gamma_2, \mathbb{P}_{\gamma_1}(a)}$. Finally,

$$\mathbb{P}_{\gamma_1 \between_\xi \gamma_2}(a) = P_{\gamma_1 \between_\xi \gamma_2, a}(1) = P_{\gamma_2, \mathbb{P}_{\gamma_1}(a)}(1) = \left(\mathbb{P}_{\gamma_2} \circ \mathbb{P}_{\gamma_1}\right)(a) \tag{8.67}$$

follows.

Subproof (5. Smooth path dependence). Let $U \in \mathcal{O}_M$ be an open set of M and let $\Psi \in C^\infty(TU, U)$ be a smooth map satisfying $\forall p \in U\colon \Psi(0_{T_pM}) = p$. Note that for each $X \in TU$, the map $\gamma_X\colon [0,1] \to U, t \mapsto \Psi(tX)$ is a smooth path starting at $\pi_{TM}(X)$.

Let $X \in TU$ and $t \in [0,1]$. Let $V \in \mathcal{O}_M$ be a neighbourhood of $\Psi(tX)$, $b_1, \ldots, b_k$ be a local frame of E on V and $e_1, \ldots, e_{\mathrm{Dim}(M)}$ be a local frame of TM on V. Denote by $\alpha \in \mathcal{B}$ the vector bundle chart associated to the local frame $b_1, \ldots, b_k$. There exists a neighbourhood $W \in \mathcal{O}_{TU}$ of X together with a constant $\epsilon > 0$ such that

$$\forall Y \in W\colon \forall s \in (t - \epsilon, t + \epsilon)\colon \Psi(sY) \subseteq V. \tag{8.68}$$

Then, for each $Y \in W$ we have the differential equation on the interval $(t-\epsilon, t+\epsilon)$

$$\xi^{m\prime}(s) + \left({\Gamma_{(e,b)}}^m{}_{ni} \circ \gamma_Y\right)(s) \cdot \xi^n(s) \cdot e^i\left(\dot{\gamma}_Y(s)\right) = 0 \tag{8.69}$$

with variable constants depending smoothly on Y. Due to the smooth dependence on the initial conditions and parameters[8] (here $Y \in W$), given some initial condition, e.g. $\xi^m(t) = c^m$, the result $\xi(s) \in \mathbb{R}^k$ for some $s \in (t-\epsilon, t+\epsilon)$ smoothly depends on $Y \in TU$ and c^m, as required. We can do this process for every $t \in [0,1]$. Hence there exists an open cover $\mathcal{V}$ of M that covers the trajectory of $\gamma_X\colon [0,1] \to M$ with the property that for every $V \in \mathcal{V}$ there exists a local frame $b_1, \ldots, b_k$ of E over V and a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ of TM over V. Since the closed interval $[0,1]$ is compact, there exists a finite subcover of $\mathcal{V}$ that does the job. Using this subcover, we can glue together the individual intervals $(t-\epsilon, t+\epsilon)$ in order to conclude that $\mathbb{P}_{\gamma_X}(a)$ depends smoothly on $(X,a) \in TU \oplus E|_U$.

Subproof (7. Initial uniqueness). Let $\gamma\colon [0,1] \to M$ and $\delta\colon [0,1] \to M$ be two smooth paths such that $\dot{\gamma}(0) = \dot{\delta}(0)$ and let $a \in E_{\gamma(0)}$. We want to show that $\dot{P}_{\gamma,a}(0) = \dot{P}_{\delta,a}(0)$.

Note that $\pi_{*,a}\dot{P}_{\gamma,a}(0) = \dot{\gamma}(0) = \dot{\delta}(0) = \pi_{*,a}\dot{P}_{\delta,a}(0)$ since $P_{\gamma,a}$ and $P_{\delta,a}$ are sections along γ and δ, respectively. Again, let $U \in \mathcal{O}_M$ be a neighbourhood of $\gamma(0) = \delta(0)$ together with a local frame $b_1, \ldots, b_k$ of E over U and a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ of TM over U. Denote by $\alpha \in \mathcal{B}$ the vector bundle chart associated to the local frame $b_1, \ldots, b_k$.

Then the section $P_{\gamma,a}$ along γ satisfies the equation

$$\mathrm{d}\alpha^m\left(\dot{P}_{\gamma,a}(0)\right) + \left({\Gamma_{(e,b)}}^m{}_{ni} \circ \gamma\right)(0) \cdot \alpha^n(a) \cdot e^i\left(\dot{\gamma}(0)\right) = 0. \tag{8.70}$$

Analogously, the section $P_{\delta,a}$ along δ satisfies the equation

$$\mathrm{d}\alpha^m\left(\dot{P}_{\delta,a}(0)\right) + \left({\Gamma_{(e,b)}}^m{}_{ni} \circ \delta\right)(0) \cdot \alpha^n(a) \cdot e^i\left(\dot{\delta}(0)\right) = 0. \tag{8.71}$$

Now, since $\dot{\gamma}(0) = \dot{\delta}(0)$, it follows that $\mathrm{d}\alpha^m\left(\dot{P}_{\gamma,a}(0)\right) = \mathrm{d}\alpha^m\left(\dot{P}_{\delta,a}(0)\right)$. However, $\dot{P}_{\gamma,a}(0)$ and $\dot{P}_{\delta,a}(0)$ lie in the same fibre T_aE of $\pi_{TE}\colon TE \to E$. Finally,

$$\dot{P}_{\gamma,a}(0) = \dot{P}_{\delta,a}(0), \tag{8.72}$$

as desired.

Subproof ($Y \in \Gamma_\gamma(E)$ parallelly transported iff $\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0$). Suppose $Y \in \Gamma_\gamma(E)$ is parallelly transported along a curve $\gamma\colon (\tau_1, \tau_2) \to M$. By definition, for every $t_1, t_2 \in (\tau_1, \tau_2)$ with $t_1 < t_2$ it holds that

$$Y(t_2) = \mathbb{P}_{\gamma_{t_1 \to t_2}}\left(Y(t_1)\right), \tag{8.73}$$

where we defined the smooth path $\gamma_{t_1\to t_2}\colon [0,1] \to M, s \mapsto \gamma\left(t_1 + s\left(t_2 - t_1\right)\right)$. However, by definition of $\mathbb{P}$, the section $P_{\gamma_{t_1\to t_2}, Y(t_1)}$ along $\gamma_{t_1\to t_2}$ satisfies $\nabla^{\gamma_{t_1\to t_2}}_{\partial_{\mathrm{id}}} P_{\gamma_{t_1\to t_2}, Y(t_1)} = 0$. Now, using the diffeomorphism $\mu\colon [t_1, t_2] \to [0,1], t \mapsto \frac{t-t_1}{t_2-t_1}$ we can establish that on the interval $[t_1, t_2]$ it holds that

$$Y|_{[t_1,t_2]} = P_{\gamma_{t_1\to t_2}, Y(t_1)} \circ \mu. \tag{8.74}$$

Remember that $P_{\gamma_{t_1\to t_2}}$ satisfies $\nabla^{\gamma_{t_1\to t_2}}_{\partial_{\mathrm{id}}} P_{\gamma_{t_1\to t_2}, Y(t_1)} = 0$. Due to reparametrization invariance, we have that $\nabla^{\gamma|_{[t_1,t_2]}}_{\partial_{\mathrm{id}}}\left(P_{\gamma_{t_1\to t_2}, Y(t_1)} \circ \mu\right) = 0$. However, we have previously seen that $Y|_{[t_1,t_2]} = P_{\gamma_{t_1\to t_2}, Y(t_1)} \circ \mu$. Substituting this expression leads us to the conclusion that on the interval $[t_1, t_2]$ it holds that

$$\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0. \tag{8.75}$$

Since $t_1 < t_2$ were arbitrary, the conclusion is true for the entire interval (τ_1, τ_2). This concludes the proof that if $Y \in \Gamma_\gamma(E)$ is parallelly transported, then $\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0$.

[8] *Ordinary Differential Equations* by V. I. Arnol'd, Page 97, Chapter 2, Section 7, Subsection 5

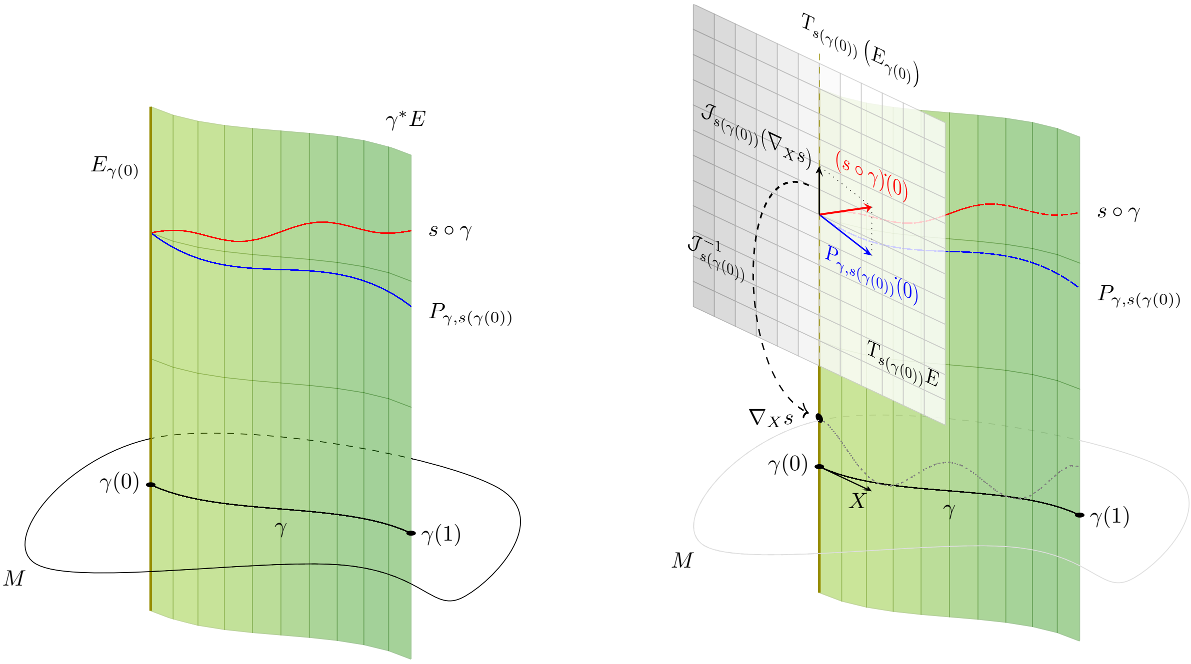

Figure 8.2: Construction of the covariant derivative operator induced by a parallel transport system.

The converse is satisfied by construction of the parallel transport system $\mathbb{P}$.

Theorem 8.9. *Let $\pi\colon E\to M$ be a vector bundle of rank k equipped with a parallel transport system $\mathbb{P}$.*

We can canonically construct a covariant derivative operator ∇ on $\pi\colon E\to M$ such that a section Y along a curve $\gamma\colon(t_1,t_2)\to M$ is parallelly transported along γ if and only if $\nabla^{\gamma}_{\partial_{\mathrm{id}}}Y = 0_{\Gamma_\gamma(E)}$.

Proof 8.9. We define:

$$\nabla\colon TM\times\Gamma(E)\to E,\quad (X,s)\mapsto \mathcal{J}^{-1}_{s(\pi_{TM}(X))}\left(s_*[X]-L_{s(\pi_{TM}(X))}(X)\right), \tag{8.76}$$

where $\mathcal{J}_{s(\pi_{TM}(X))}\colon E_{\pi_{TM}(X)}\to T_a(E_{\pi_{TM}(X)})$ is the canonical isomorphism, and $L_{s(\pi_{TM}(X))}\colon T_{\pi_{TM}(X)}M\to T_{s(\pi_{TM}(X))}E$ is the linear map from lemma 8.6. We can apply $\mathcal{J}^{-1}_{s(\pi_{TM}(X))}$ to $s_*[X]-L_{s(\pi_{TM}(X))}(X)\in T_{s(\pi_{TM}(X))}E$ due to the fact that it is true that $\pi_*\circ s_*(X)=X$ and $\pi_*\circ L_{s(\pi_{TM}(X))}=X$, see the last part of the proof of lemma 8.6. This guarantees that $s_*[X]-L_{s(\pi_{TM}(X))}(X)$ is an element of the subspace $T_{s(\pi_{TM}(X))}\left(E_{\pi_{TM}(X)}\right)$, ensuring that we can apply $\mathcal{J}^{-1}_{s(\pi_{TM}(X))}$ in order to obtain an element in the fibre $E_{\pi_{TM}(X)}$. Refer to fig. 8.2 for an exemplary illustration.

Observe that $\nabla_X s\in E_{\pi_{TM}(X)}$, so if we are given a vector field $X\in\Gamma(TM)$ instead, then the map $\nabla_X s\colon p\mapsto\nabla_{X_p}s$ defines a global section of $\pi\colon E\to M$. What we need to prove is that $\nabla_X s$ is smooth, we will check this towards the end of this proof.

First, however, let us check items 1 to 4 from definition 8.1. We start with items 1 and 2.

Subproof (Items 1 and 2). This is a corollary of lemma 8.6. The differential $s_{*,\pi_{TM}(X)}\colon T_{\pi_{TM}(X)}M\to T_{s(\pi_{TM}(X))}E$, the map $L_{s(\pi_{TM}(X))}\colon T_{\pi_{TM}(X)}M\to T_{s(\pi_{TM}(X))}E$ from lemma 8.6 and the inverse of the canonical isomorphism $\mathcal{J}_{s(\pi_{TM}(X))}\colon T_{s(\pi_{TM}(X))}\left(E_{\pi_{TM}(X)}\right)\to E_{\pi_{TM}(X)}$ are linear maps.

Before moving on to items 3 and 4, it is worth to rewrite $\nabla_X s$ in terms of a smooth path $\gamma_X \colon [0,1] \to M$ that satisfies $\dot{\gamma}_X(0) = X$:

$$\nabla \colon TM \times \Gamma(E) \to E, \quad (X, s) \mapsto \mathcal{J}^{-1}_{s(\pi_{TM}(X))} \left((s \circ \gamma_X)\dot{}(0) - P_{\gamma_X, s(\pi_{TM}(X))}\dot{}(0) \right). \tag{8.77}$$

We can arrive at an even more useful expression by invoking lemma 8.7 given a vector bundle chart α of $\pi \colon E \to M$. Then, for every $X \in \pi^{-1}_{TM}[\pi[\mathrm{Dom}(\alpha)]]$ it holds that

$$\nabla_X(s) = (\pi, \alpha)^{-1} \Big(\pi_{TM}(X), (\alpha \circ s \circ \gamma_X)'(0) - \left(\alpha \circ P_{\gamma_X, s(\pi_{TM}(X))}\right)'(0) \Big). \tag{8.78}$$

This relation is particularly useful because we can directly make use of the addition of curves in $\mathbb{R}^k$.

Subproof (Item 3). Suppose we are given two sections $s_1, s_2 \in \Gamma(E)$. Then:

$$\nabla_X(s_1 + s_2) = (\pi, \alpha)^{-1} \Big(\pi(X), \left(\alpha \circ (s_1 + s_2) \circ \gamma_X\right)'(0) - \left(\alpha \circ P_{\gamma_X, s_1(\pi(X)) + s_2(\pi(X))}\right)'(0) \Big), \tag{8.79}$$

where we have abusively abbreviated π_{TM} by π and omitted the indices of the operations $+_{\Gamma(E)}$ and $+_{E_{\pi(X)}}$. First, note that for every $t \in [0,1]$, the value $P_{\gamma_X, s_1(\pi(X))+s_2(\pi(X))}(t)$ is equal to the sum $P_{\gamma_X, s_1(\pi(X))}(t) + P_{\gamma_X, s_2(\pi(X))}(t)$, due to item 1 from definition 8.7. Subsequently, we can use the fibre-wise linearity of the vector bundle chart α and calculus in $\mathbb{R}^k$ in order to deduce that

$$\left(\alpha \circ (s_1 + s_2) \circ \gamma_X\right)'(0) = (\alpha \circ s_1 \circ \gamma_X)'(0) + (\alpha \circ s_2 \circ \gamma_X)'(0). \tag{8.80}$$

In just the same fashion, we arrive at

$$\left(\alpha \circ \left(P_{\gamma_X, s_1(\pi(X))} + P_{\gamma_X, s_2(\pi(X))}\right)\right)'(0) = \left(\alpha \circ P_{\gamma_X, s_1(\pi(X))}\right)'(0) + \left(\alpha \circ P_{\gamma_X, s_2(\pi(X))}\right)'(0). \tag{8.81}$$

Substituting these results in equation (8.79), and using once more the fibre-wise linearity of the vector bundle chart α in order to arrive at the desired result

$$\nabla_X \left(s_1 \underset{\Gamma(E)}{+} s_2 \right) = \nabla_X s_1 \underset{E_{\pi(X)}}{+} \nabla_X s_2. \tag{8.82}$$

Subproof (Item 4). Suppose we are given a section $s \in \Gamma(E)$ and a smooth function $\varphi \in C^\infty(M)$. Then:

$$\nabla_X(\varphi \cdot s) = (\pi, \alpha)^{-1} \Big(\pi(X), \left(\alpha \circ (\varphi \cdot s) \circ \gamma_X\right)'(0) - \left(\alpha \circ P_{\gamma_X, \varphi(\pi(X)) \cdot s(\pi(X))}\right)'(0) \Big), \tag{8.83}$$

The proof is very similar to the one of the above item. First, note that for every $t \in [0,1]$, the value $P_{\gamma_X, \varphi(\pi(X)) \cdot s(\pi(X))}(t)$ is equal to $\varphi(\pi(X)) \cdot P_{\gamma_X, s(\pi(X))}(t)$, due to item 1 from definition 8.7. Subsequently, we can use the fibre-wise linearity of the vector bundle chart α and calculus in $\mathbb{R}^k$ in order to deduce that

$$(\alpha \circ (\varphi \cdot s) \circ \gamma_X)'(0) = (\varphi \circ \gamma_X)'(0) \cdot \alpha(s(\gamma_X(0))) + \varphi(\gamma_X(0)) \cdot (\alpha \circ s \circ \gamma_X)'(0) \tag{8.84}$$

and

$$\left(\alpha \circ \left(\varphi(\pi(X)) \cdot P_{\gamma_X, s(\pi(X))}\right)\right)'(0) = \varphi(\pi(X)) \cdot \left(\alpha \circ P_{\gamma_X, s(\pi(X))}\right)'(0). \tag{8.85}$$

Substituting these results in equation (8.83), and using once more the fibre-wise linearity of the vector bundle chart α in order to arrive at the desired result

$$\nabla_X \left(\varphi \underset{\Gamma(E)}{\cdot} s \right) = X(\varphi) \underset{E_{\pi(X)}}{\cdot} s(\pi(X)) \underset{E_{\pi(X)}}{+} \varphi(\pi(X)) \underset{E_{\pi(X)}}{\cdot} \nabla_X s. \tag{8.86}$$

Subproof (Smoothness). Let now $X \in \Gamma(TM)$ be a vector field. We want to show that the map

$$\nabla_X s \colon M \to E, \quad p \mapsto \nabla_{X_p} s \tag{8.87}$$

is smooth. Let $p \in M$. Suppose $x \in \mathcal{A}_M$ is a chart of M at p such that $x[\mathrm{Dom}(x)] = \mathbb{R}^{\mathrm{Dim}(M)}$ and consider the map $\Psi^{(x)} \colon T\,\mathrm{Dom}(x) \to \mathrm{Dom}(x)$ from lemma 8.4 that satisfies the hypothesis of item 5 from definition 8.7. For every $X \in T\,\mathrm{Dom}(x)$, define the smooth path (see lemma 8.4)

$$\gamma_X^{(x)} \colon [0,1] \to M, \quad t \mapsto \Psi^{(x)}(tX). \tag{8.88}$$

Let $\alpha \colon \mathrm{Dom}(\alpha) \to \mathbb{R}^k$ be a vector bundle chart at p. Without loss of generality, assume that x was chosen such that $\mathrm{Dom}(x) \subseteq \pi[\mathrm{Dom}(\alpha)]$. Then, for every $Z \in T\,\mathrm{Dom}(x)$ it holds that

$$\nabla_Z s = (\pi, \alpha)^{-1} \left(\pi_{TM}(Z), (\alpha_* \circ s_*)[Z] - \left(\alpha \circ P_{\gamma_Z^{(x)}, s(\pi_{TM}(Z))} \right)' (0) \right). \tag{8.89}$$

The minuend $(\alpha_* \circ s_*)[Z]$ depends smoothly on Z, since the differentials $s_* \colon TM \to TE$ and $\alpha_* \colon T\,\mathrm{Dom}(\alpha) \to \mathbb{R}^k$ are smooth maps. The subtrahend $\left(\alpha \circ P_{\gamma_Z^{(x)}, s(\pi_{TM}(Z))} \right)' (0)$ also depends smoothly on Z, this, however, is less obvious to realize.
First, note that we can write

$$\left(\alpha \circ P_{\gamma_Z^{(x)}, s(\pi_{TM}(Z))} \right)' (0) = \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} \left(\alpha \circ \mathbb{P}_{r \mapsto \Psi^{(x)}(rtZ)} \left(s(\pi_{TM}(Z)) \right) \right). \tag{8.90}$$

The expression inside the brackets is checked to depend smoothly on $Z \in T\,\mathrm{Dom}(x)$ and $t \in \mathbb{R}$ thanks to item 5 from definition 8.7, once we introduce the auxiliary smooth map

$$T\,\mathrm{Dom}(x) \times [0,1] \to T\,\mathrm{Dom}(x) \oplus E|_{\mathrm{Dom}(x)}, \quad (Z, t) \mapsto (tZ, s(\pi_{TM}(Z))). \tag{8.91}$$

Then, however, its derivative with respect to t at $t = 0$ still depends smoothly on $Z \in T\,\mathrm{Dom}(x)$. Altogether, this establishes that $\nabla_Z s$ depends smoothly on $Z \in T\,\mathrm{Dom}(x)$. As a consequence, given the vector field $X \in \Gamma(TM)$ the map

$$\nabla_X s \colon M \to E, \quad p \mapsto \nabla_{X_p} s \tag{8.92}$$

is smooth.

Subproof ($Y \in \Gamma_\gamma(E)$ parallelly transported iff $\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0$). Let $\gamma \colon (\tau_1, \tau_2) \to M$ be a curve in M. Suppose we are given a section $Y \in \Gamma_\gamma(E)$ along γ that satisfies $\nabla^\gamma_{\partial_{\mathrm{id}}} Y = 0_{\Gamma_\gamma(E)}$. Fix $t_1, t_2 \in (\tau_1, \tau_2)$ such that $t_1 < t_2$ and define the smooth path

$$\gamma_{t_1 \to t_2} \colon [0,1] \to M, \quad s \mapsto \gamma(t_1 + s(t_2 - t_1)). \tag{8.93}$$

Let $s \in [0,1]$ and let α be a vector bundle chart at $\gamma_{t_1 \to t_2}(s)$. There exits $\epsilon > 0$ such that $\gamma_{t_1 \to t_2}[(s - \epsilon, s + \epsilon)] \subseteq \pi[\mathrm{Dom}(\alpha)]$. By definition of ∇, we have that for any $r \in (s - \epsilon, s + \epsilon)$ it holds that

$$0_{E_{\gamma_{t_1 \to t_2}(r)}} = \left(\nabla^\gamma_{\partial_{\mathrm{id}}} Y \right)(\mu(r)) = (\pi, \alpha)^{-1} \left(\gamma(r), (\alpha \circ Y \circ \mu)'(r) - \left(\alpha \circ P_{\gamma_{t_1 \to t_2}, Y(t_1)} \right)' (r) \right), \tag{8.94}$$

where we made use of the diffeomorphism $\mu \colon [0,1] \to [t_1, t_2], r \mapsto t_1 + (t_2 - t_1) r$. Since α is an isomorphism on the fibres, we have

$$(\alpha \circ Y \circ \mu)'(r) = \left(\alpha \circ P_{\gamma_{t_1 \to t_2}, Y(t_1)} \right)' (r). \tag{8.95}$$

The ordinary differential equation implies that there exists $C_s \in \mathbb{R}^k$ such that

$$\alpha \circ Y \circ \mu(r) = \alpha \circ P_{\gamma_{t_1 \to t_2}, Y(t_1)}(r) + C_s. \tag{8.96}$$

Since the closed interval $[0, 1]$ is compact, we can repeat the same procedure on a finite number of such open intervals that cover $[0, 1]$. Then, however, $Y(t_1) = Y(\mu(0)) = P_{\gamma_{t_1 \to t_2}, Y(t_1)}(0)$ implies that the constants C_s all vanish. Consequently,

$$Y(t_2) = Y(\mu(1)) = P_{\gamma_{t_1 \to t_2}, Y(t_1)}(1) = \mathbb{P}_{\gamma_{t_1 \to t_2}}(Y(t_1)). \tag{8.97}$$

This proves that Y is parallelly transported along γ.

The converse is true by construction of ∇. This concludes the proof.

Remark 8.3. The concepts of a covariant derivative operator and a parallel transport system are equivalent to what is referred to as a *linear connection on a vector bundle*. The latter concept will not be introduced here. However, it is common terminology to also refer to the equivalent concepts of covariant derivative operators and parallel transport systems by the term *linear connections*. Technically, the third definition of a *linear connection* that we omit here makes it easier to relate covariant derivatives with their principal bundle analogon, simply called *principal bundle connections*, which enable us to talk about *curvature* on a principal bundle. This will not be covered in this work. We will instead restrict ourselves to the study of *curvature* of covariant derivative operators on vector bundles for reasons of brevity and practicability.

Remark 8.4. Unlike *linear connections* and covariant derivative operators, which can only be defined on vector bundles, the concept of parallel transport systems may be generalized to affine bundles.

Sometimes *linear connections* are referred to as *affine connections*, refer to remark 8.1. This is misleading, because it is assumed that they are defined on vector bundles.

Definition 8.9. Let $\pi\colon E \to M$ be a vector bundle equipped with a covariant derivative operator ∇.

There is a canonical way to extend the definition of ∇ to the tensor product bundles of $\pi\colon E \to M$ and its dual bundle $\pi^*\colon E^* \to M$. Beyond the items 1 to 4 from definition 8.1, we require two additional conditions in order to uniquely define ∇ on each (r, s)-tensor product bundle.

5. For every two tensor fields $S \in \Gamma(E^{(m,n)})$ and $T \in \Gamma(E^{(r,s)})$, it holds that

$$\forall X \in \Gamma(TM)\colon \quad \nabla_X(S \otimes T) = \nabla_X S \otimes T + S \otimes \nabla_X T, \tag{8.98}$$

where $S \otimes T \in \Gamma(E^{(m+r,n+s)})$ is the tensor product of S and T.

6. For every tensor field $S \in \Gamma(E^{(m,n)})$ with $m, n \geq 1$ it holds that

$$\forall X \in \Gamma(TM)\colon \quad C^m_n\left(\nabla_X S\right) = \nabla_X\left(C^m_n(S)\right), \tag{8.99}$$

where $C^m_n(S) \in \Gamma(E^{(m-1,n-1)})$ is the (m, n)-contraction of S.

7. On the $(0, 0)$-tensor product bundle, also known the algebra of smooth functions $C^\infty(M)$, the covariant derivative operator ∇ coincides with the differential d. That is, for every smooth function $f \in C^\infty(M)$ it holds that

$$\forall X \in \Gamma(TM)\colon \quad \nabla_X f = (\mathrm{d}f)(X) = X(f). \tag{8.100}$$

For reasons of brevity, we will omit the indices of the operators $+$, $\cdot$ and $\otimes$ for the remainder of this chapter. It should be clear from the context which operation is meant. It is left as an exercise to the reader to trace the indices of the operations at play.

Lemma 8.10. *Definition 8.9 consistently defines a covariant derivative operator ∇ on the dual bundle $\pi^*\colon E^* \to M$.*

Proof 8.10. For every section $\omega \in \Gamma(E^*)$ of the dual bundle $\pi^*\colon E \to M$, every section $Y \in \Gamma(E)$ of the original bundle $\pi\colon E \to M$ and every vector field $X \in \Gamma(TM)$, it holds that

$$\nabla_X\left(C^1_1\left(\omega \otimes Y\right)\right) = C^1_1\left(\nabla_X\left(\omega \otimes Y\right)\right), \tag{8.101}$$

$$= C^1_1\left(\nabla_X \omega \otimes Y + \omega \otimes \nabla_X Y\right), \tag{8.102}$$

where we used items 5 and 6 from definition 8.9. The left hand side can be rewritten as $\nabla_X(\omega(Y))$ where $\omega(Y) \in C^\infty(M)$ is a smooth function. By item 7 from definition 8.9, however, the left hand side can be written as $X(\omega(Y))$, where the vector field $X \in \Gamma(TM)$ derives the smooth function $\omega(Y)$. The right hand side, on the other hand, can be rewritten as $(\nabla_X\omega)(Y) + \omega(\nabla_X Y)$. Altogether, this provides a definition of ∇ on $\Gamma(E^*)$:

$$(\nabla_X\omega)(Y) = X(\omega(Y)) - \omega(\nabla_X Y). \tag{8.103}$$

Indeed, the above equation defines a section $\nabla_X\omega \in \Gamma(E^*)$: First, it is smooth due to the fact that the right hand side is a smooth function for any choice of $X \in \Gamma(TM)$ and $Y \in \Gamma(E)$. Second, for every two sections $Y_1, Y_2 \in \Gamma(E)$, it holds that

$$\begin{aligned}
(\nabla_X\omega)(Y_1 + Y_1) &= X(\omega(Y_1 + Y_2)) - \omega(\nabla_X(Y_1 + Y_2)), &(8.104)\\
&= X(\omega(Y_1)) + X(\omega(Y_2)) - \omega(\nabla_X(Y_1)) - \omega(\nabla_X(Y_2)), &(8.105)\\
&= (\nabla_X\omega)(Y_1) + (\nabla_X\omega)(Y_2), &(8.106)
\end{aligned}$$

where we used item 3 from definition 8.1 for ∇ on $\pi\colon E \to M$. Third, for every section $Y \in \Gamma(E)$ and every smooth function $\varphi \in C^\infty(M)$, it holds that

$$\begin{aligned}
(\nabla_X\omega)(\varphi \cdot Y) &= X(\omega(\varphi \cdot Y)) - \omega(\nabla_X(\varphi \cdot Y)), &(8.107)\\
&= X(\varphi) \cdot \omega(Y) + \varphi \cdot X(\omega(Y)) - \omega(X(\varphi)Y - \varphi \cdot \nabla_X Y), &(8.108)\\
&= \varphi \cdot (\nabla_X\omega)(Y), &(8.109)
\end{aligned}$$

where we used item 4 from definition 8.1 for ∇ on $\pi\colon E \to M$. This concludes the proof that $\nabla_X\omega$ is indeed a section of the dual bundle $\pi^*\colon E^* \to M$ by means of proposition 6.1.

It is left to show that ∇ satisfies the items 1 to 4 from definition 8.1 on the dual bundle $\pi^*\colon E^* \to M$. This can be directly read off from (8.103), bearing in mind that ∇ satisfies items 1 and 2 from definition 8.1 on $\pi\colon E \to M$. This concludes the proof that ∇ is a covariant derivative operator on $\pi^*\colon E^* \to M$.

Lemma 8.11. *Definition 8.9 consistently defines a covariant derivative operator ∇ on the (r,s)-tensor product bundle $\pi^{(r,s)}\colon E^{(r,s)} \to M$.*

Proof 8.11. Let $T \in \Gamma(E^{(r,s)})$ be a tensor field, $X \in \Gamma(TM)$ a vector field, $Y_1, \dots, Y_s \in \Gamma(E)$ sections of $\pi\colon E \to M$, and $\omega_1, \dots, \omega_r \in \Gamma(E^*)$ sections of the dual bundle $\pi\colon E^* \to M$. Then:

$$\begin{aligned}
\nabla_X(T(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s)) = &(\nabla_X T)(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s)\\
&+ T(\nabla_X\omega_1, \dots, \omega_r, Y_1, \dots, Y_s) + \dots + T(\omega_1, \dots, \nabla_X\omega_r, Y_1, \dots, Y_s)\\
&+ T(\omega_1, \dots, \omega_r, \nabla_X Y_1, \dots, Y_s) + \dots + T(\omega_1, \dots, \omega_r, Y_1, \dots, \nabla_X Y_s),
\end{aligned} \tag{8.110}$$

where we used items 5 and 6 from definition 8.9. By item 7 from definition 8.9, the left hand side can be written as $X(T(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s))$, where the vector field $X \in \Gamma(TM)$ derives the smooth function $T(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s) \in C^\infty(M)$. Using this relation, we provide a definition of ∇ on $\Gamma(E^{(r,s)})$:

$$\begin{aligned}
(\nabla_X T)(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s) = &X(T(\omega_1, \dots, \omega_r, Y_1, \dots, Y_s))\\
&- T(\nabla_X\omega_1, \dots, \omega_r, Y_1, \dots, Y_s) - \dots - T(\omega_1, \dots, \nabla_X\omega_r, Y_1, \dots, Y_s)\\
&- T(\omega_1, \dots, \omega_r, \nabla_X Y_1, \dots, Y_s) - \dots - T(\omega_1, \dots, \omega_r, Y_1, \dots, \nabla_X Y_s).
\end{aligned} \tag{8.111}$$

Observe that the right hand side defines a smooth function on M. It is left as an exercise to the reader to check that $\nabla_X T$ is $C^\infty(M)$-linear in all its arguments. By proposition 6.1, this proves that $\nabla_X T$ is indeed a section of the (r,s)-tensor product bundle $\pi^{(r,s)}\colon E^{(r,s)} \to M$.

It is now quickly verified that ∇ satisfies the items 1 to 4 from definition 8.1 on the (r,s)-tensor product bundle $\pi^{(r,s)}\colon E^{(r,s)} \to M$. This concludes the proof that ∇ is a covariant derivative operator on $\pi^{(r,s)}\colon E^{(r,s)} \to M$.

Lemma 8.12. *Suppose we are given an open set $U \in \mathcal{O}_M$ together with a local frame $b_1, \ldots, b_k \in \Gamma(E|_U)$ of $\pi\colon E \to M$ and a local frame $e_1, \ldots, e_{\mathrm{Dim}(M)} \in \Gamma(TU)$ of $\pi_{TM}\colon TM \to M$. Denote by $b^1, \ldots, b^k \in \Gamma(E^*|_U)$ the dual frame of $e_1, \ldots, e_k$.*

The coefficient functions ${\Gamma^*_{(e,b)}}^m{}_{ni}$ of the covariant derivative operator ∇ on the dual bundle $\pi*\colon E^* \to M$ with respect to the local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ for TM over U and the dual frame $b^1, \ldots, b^k$ for E over U satisfy

$$
{\Gamma^*_{(e,b)}}^m{}_{ni} = -{\Gamma_{(e,b)}}^n{}_{mi}. \tag{8.112}
$$

Proof 8.12. First, note that from the perspective of $\pi^*\colon E^* \to M$, the dual frame $b^1, \ldots, b^k \in \Gamma(E^*|_U)$ of $e_1, \ldots, e_k \in \Gamma(E|_U)$ is actually just a local frame. This is the reason why the indices appear to be in the wrong places, once we write down the definition of the coefficient functions ${\Gamma^*_{(e,b)}}^m{}_{ni}$:

$$
{\Gamma^*_{(e,b)}}^m{}_{ni} = \left(\nabla_{e_i} b^n\right)(b_m). \tag{8.113}
$$

The latter can be evaluated using (8.103). We find

$$
\left(\nabla_{e_i} b^n\right)(b_m) = e_i\left(b^n(b_m)\right) - b^n\left(\nabla_{e_i} b_m\right). \tag{8.114}
$$

The first term on the right hand side vanishes due to the fact that $b^n(b_m) = \delta^n_m$ is constant on U. The second term matches (up to the sign) the definition of the coefficient functions ${\Gamma_{(e,b)}}^n{}_{mi}$ of ∇ on $\pi\colon E \to M$ with respect to the local frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$ for TM over U and the original frame $b_1, \ldots, b_k$ for E over U. This leads us to the desired result

$$
{\Gamma^*_{(e,b)}}^m{}_{ni} = -{\Gamma_{(e,b)}}^n{}_{mi}. \tag{8.115}
$$

Corollary 8.13. *For every vector field $X \in \Gamma(TM)$ and every section $s \in \Gamma(E^*)$ of the dual bundle $\pi^*\colon E^* \to M$ it holds on U that*

$$
\nabla_X \omega = \left(X(\omega^{(b)}_m) - {\Gamma_{(e,b)}}^n{}_{mj} X^j_{(e)} \omega^{(b)}_n\right) b^m. \tag{8.116}
$$

Remark 8.5. Equation (8.112) from lemma 8.12 makes sure that the coefficient functions of ∇ on the dual bundle $\pi^*\colon E^* \to M$ satisfy the transformation formula (8.22) from exercise 8.2 in their own right. This is easy to observe: In order to change into the perspective from $\pi^*\colon E^* \to M$, all we have to do is change the location of indices, exchanging the notation of ${B^m}_n$ with the one of ${B_m}^n$. Together with making use of the fact that $B^k_m B^n_k = \delta^n_m$, this shows that equation (8.22) continues true for ${\Gamma^*_{(e,b)}}^m{}_{nj}$, as required.

This is merely a consistency check that we have done everything right. Working this out in detail, however, requires us to exercise caution from the conceptual point of view in order not to confuse ourselves. As such, it is a worthy exercise.

Definition 8.10. Let $\pi\colon E \to M$ be a vector bundle of rank k equipped with a bundle metric g.

A covariant derivative operator ∇ on $\pi\colon E \to M$ is said to be **metric-compatible with respect to** g, or just g**-compatible**, if for every vector field $X \in \Gamma(TM)$ it holds that

$$
\nabla_X g = 0. \tag{8.117}
$$

Theorem 8.14. *Let $\pi\colon E \to M$ be a vector bundle of rank k equipped with a bundle metric g and a metric-compatible covariant derivative operator ∇.*

Let $\gamma\colon (\tau_1, \tau_2) \to M$ be a curve. For every two parallelly transported sections $Y, Z \in \Gamma_\gamma(E)$ along γ, it holds that

$$
\nabla^\gamma_{\partial_{\mathrm{id}}}\left(\left(g \circ \gamma\right)(Y, Z)\right) = 0. \tag{8.118}
$$

That is, $\left(\gamma^* g\right)(Y, Z)$ is constant along γ.

Proof 8.14. Using items 5 and 6 from definition 8.9, we deduce

$$\begin{aligned}\nabla^{\gamma}_{\partial_{\mathrm{id}}}\left((g\circ\gamma)(Y,Z)\right) &= \left(\nabla^{\gamma}_{\partial_{\mathrm{id}}}(g\circ\gamma)\right)(Y,Z) + (g\circ\gamma)\left(\nabla^{\gamma}_{\partial_{\mathrm{id}}}Y,Z\right) + (g\circ\gamma)\left(Y,\nabla^{\gamma}_{\partial_{\mathrm{id}}}Z\right)\\ &= 0,\end{aligned} \tag{8.119}$$

Here we used the hypothesis that $\nabla^{\gamma}_{\partial_{\mathrm{id}}}Y = 0$ and $\nabla^{\gamma}_{\partial_{\mathrm{id}}}Z = 0$ (Y and Z are parallelly transported along γ) and the metric-compatibility condition (8.117), which in terms of $\nabla^{\gamma}_{\partial_{\mathrm{id}}}$ is expressed as

$$\left(\nabla^{\gamma}_{\partial_{\mathrm{id}}}(g\circ\gamma)\right)(t) = \nabla_{\dot{\gamma}(t)}g = 0, \tag{8.120}$$

using equation (8.23) from proposition 8.2.

Corollary 8.15. *Given two sections $Y,Z\in\Gamma(E)$ that satisfy $\nabla_X Y = 0$ and $\nabla_X Z = 0$ for every vector field $X\in\Gamma(TM)$ (or differently put, sections Y and Z whose corresponding sections $Y\circ\gamma$ and $Z\circ\gamma$ along γ are parallelly transported along any curve $\gamma\colon(\tau_1,\tau_2)\to M$), the function $g(Y,Z)\in C^{\infty}(M)$ is constant on the connected components of M.*

9 Holonomy and curvature

Definition 9.1. Let $\pi\colon E \to M$ be a vector bundle equipped with a covariant derivative operator ∇. Denote by $\mathbb{P}$ the parallel transport system induced by ∇ through theorem 8.8.

The **holonomy group of ∇ at a point** $p \in M$ is the subgroup

$$\mathrm{Hol}^{\nabla}(p) := \{\mathbb{P}_{\gamma}\colon E_p \to E_p \mid \gamma \text{ is a piecewise smooth loop based at } p\} \tag{9.1}$$

of the general linear group $\mathrm{GL}(E_p)$ over the fibre E_p.

Remark 9.1. Observe that $\mathrm{Hol}^{\nabla}(p)$ is a subgroup of $\mathrm{GL}(E_p)$. It suffices to recall item 2 from definition 8.7 and the result of lemma 8.3.

Definition 9.2. Let $\pi\colon E \to M$ be a vector bundle equipped with a covariant derivative operator ∇. Denote by $\mathbb{P}$ the parallel transport system induced by ∇ through theorem 8.8.

The **restricted holonomy group of ∇ at a point** $p \in M$ is the subgroup

$$\mathrm{Hol}_0^{\nabla}(p) := \{\mathbb{P}_{\gamma}\colon E_p \to E_p \mid \gamma \text{ is a contractible piecewise smooth loop based at } p\} \tag{9.2}$$

of the general linear group $\mathrm{GL}(E_p)$ over the fibre E_p and is naturally a subgroup of the holonomy group $\mathrm{Hol}^{\nabla}(p)$ of ∇ at p.

Proposition 9.1. *Let $\pi\colon E \to M$ be a vector bundle over M of rank k equipped with a covariant derivative operator ∇ and let $p, q \in M$ be points in the same connected component of M.*

Then the holonomy groups $\mathrm{Hol}^{\nabla}(p)$ and $\mathrm{Hol}^{\nabla}(q)$ at p and q, respectively, are isomorphic.

Proof 9.1. The smooth manifold $(M, \mathcal{O}_M, \mathcal{A}_M)$ is by definition locally Euclidean and, as such, *locally path-connected*. As a consequence, the *connected components* of M coincide with the *path-connected components* of M. Therefore, there exists a (continuous, but not necessarily smooth) path $\gamma\colon [0,1] \to M$ from $p = \gamma(0)$ to $q = \gamma(1)$. It is, however, always possible to find a smooth path $\gamma\colon [0,1] \to M$ from p to q. This is ultimately due to the Whitney embedding theorem.[9]

Define the map

$$L_{\gamma}\colon\ \mathrm{Hol}^{\nabla}(p) \to \mathrm{Hol}^{\nabla}(q), \quad \mathbb{P}_{\delta} \mapsto \mathbb{P}_{(\overline{\gamma} * \delta) * \gamma}, \tag{9.3}$$

where $\overline{\gamma}\colon [0,1] \to M$ denotes the path inversion of γ. Indeed, for every piecewise smooth loop $\delta\colon [0,1] \to M$ at p the path concatenation $(\overline{\gamma} * \delta) * \gamma\colon [0,1] \to M$ is a piecewise smooth loop at p, too. Moreover, by item 3 from definition 8.7, we have that

$$L_{\gamma}(\mathbb{P}_{\delta}) = \mathbb{P}_{\gamma} \circ \mathbb{P}_{\delta} \circ \mathbb{P}_{\overline{\gamma}}. \tag{9.4}$$

Combined with item 2 from definition 8.7, this proves that the map

$$L_{\overline{\gamma}}\colon\ \mathrm{Hol}^{\nabla}(q) \to \mathrm{Hol}^{\nabla}(p), \quad \mathbb{P}_{\epsilon} \mapsto \mathbb{P}_{(\gamma * \epsilon) * \overline{\gamma}} = \mathbb{P}_{\overline{\gamma}} \circ \mathbb{P}_{\epsilon} \circ \mathbb{P}_{\gamma} \tag{9.5}$$

is the inverse of L_{γ}.

[9] *Introduction to Smooth Manifolds* (2nd edition) by J. M. Lee, Page 141, Theorem 6.26 (Whitney Approximation Theorem)

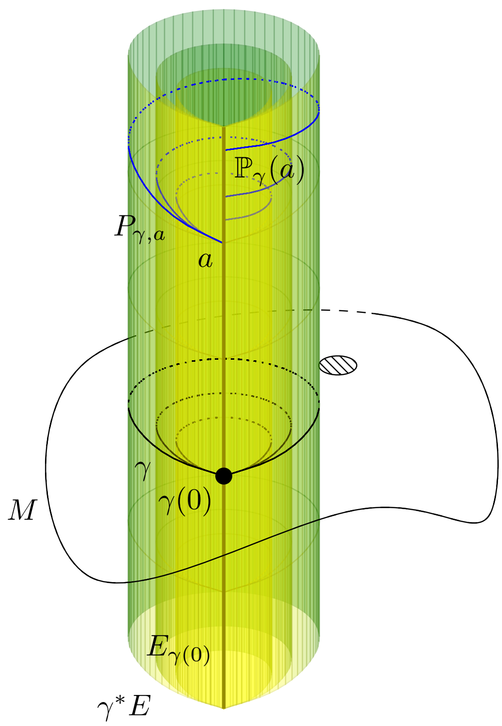

Figure 9.1: The restricted holonomy group Hol_0^∇ encompasses only the parallel transport along contractible loops.

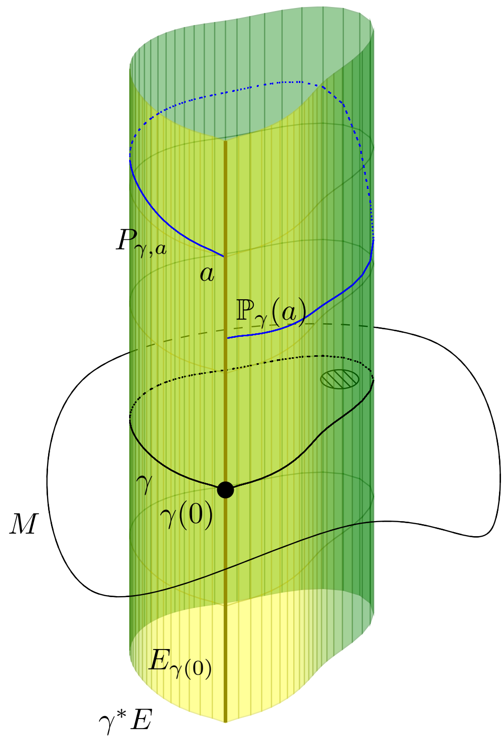

Figure 9.2: The holonomy group Hol^∇ encompasses also the parallel transport along non-contractible loops.

It is left to show that L_γ is a homomorphism and thus an isomorphism. We have:

$$L_\gamma\left(\mathbb{P}_\epsilon \circ \mathbb{P}_\delta\right) = \mathbb{P}_\gamma \circ \mathbb{P}_\epsilon \circ \mathbb{P}_\delta \circ \mathbb{P}_{\overline{\gamma}}, \tag{9.6}$$

$$= \mathbb{P}_\gamma \circ \mathbb{P}_\epsilon \circ \mathbb{P}_{\overline{\gamma}} \circ \mathbb{P}_\gamma \circ \mathbb{P}_\delta \circ \mathbb{P}_{\overline{\gamma}}, \tag{9.7}$$

$$= L_\gamma\left(\mathbb{P}_\epsilon\right) \circ L_\gamma\left(\mathbb{P}_\delta\right), \tag{9.8}$$

where we used items 2 and 3 from definition 8.7 once more.

Lemma 9.2. *Let $\pi\colon E \to M$ be a vector bundle over M of rank k equipped with a covariant derivative operator ∇ and let $p, q \in M$ be a points. Suppose $\gamma\colon [0,1] \to M$ is a piecewise smooth path from p to q.*

We can find a path homotopy $c\colon [0,1] \times [0,1] \to M$ from γ to a smooth path γ_1 with the property that for every $r \in (0,1]$, the path $\gamma_r\colon [0,1] \to M, t \mapsto c(r,t)$ is smooth. Moreover, the parallel transport $\mathbb{P}_\gamma(a)$ of a point $a \in E_p$ along γ is given by the limit

$$\mathbb{P}_\gamma(a) = \lim_{r\to 0} \mathbb{P}_{\gamma_r}(a). \tag{9.9}$$

Proof 9.2. Suppose we are given a piecewise smooth path $\gamma\colon [0,1] \to M$ with corners at the values $0 < r_1 < \cdots < r_m < 1$. For every corner value $r_i \in (0,1)$ choose a chart $x_{(i)} \in \mathcal{A}_M$ with convex domain $x[\mathrm{Dom}(x)]$. There exist numbers $\epsilon_i > 0$ such that $\gamma[[r_i - \epsilon_i, r_i + \epsilon_i]] \subset \mathrm{Dom}(x_{(i)})$. In particular, we may choose the ϵ_i small enough such that the intervals $[r_i - \epsilon_i, r_i + \epsilon_i]$ do not overlap.

Next, let us use a bump function in order to smoothen the corners out. To this end, we explicitly define for every $\delta > 0$ the bump function

$$\Psi_\delta \colon \mathbb{R} \to \mathbb{R}, \quad t \mapsto \begin{cases} \frac{1}{I_n \delta} \exp\left(-\frac{\delta^2}{\delta^2 - s^2}\right) & \text{for } t \in (-\delta, \delta), \\ 0 & \text{otherwise.} \end{cases} \tag{9.10}$$

We can choose the normalization constant I_n such that $\int_{\mathbb{R}} \Psi_\delta(\sigma)\,\mathrm{d}\sigma = 1$. Now define for every corner i and every $\delta \le \frac{\epsilon_i}{2}$ the map

$$\tilde{\gamma}_{i,\delta} \colon \left(r_i - \frac{\epsilon_i}{2}, r_i + \frac{\epsilon_i}{2}\right) \to \mathbb{R}^{\mathrm{Dim}(M)}, \quad t \mapsto \int_{-\frac{\epsilon_i}{2}}^{\frac{\epsilon_i}{2}} \Psi_{\delta \cdot \Psi_{\frac{\epsilon_i}{2}}(t - r_i)}(\sigma) \cdot (x_{(i)} \circ \gamma)(t - \sigma)\,\mathrm{d}\sigma. \tag{9.11}$$

through a modulated convolution of $(x_{(i)} \circ \gamma)$ with Ψ_δ.

Returning to the big picture, we can now define a path homotopy given by

$$c \colon [0,1] \times [0,1] \to M, \quad (\xi, t) \mapsto \begin{cases} x_{(i)}^{-1}\left(\tilde{\gamma}_{i,\xi \cdot \frac{\epsilon_i}{2}}(t)\right) & \text{for } \xi > 0 \text{ and } t \in \left(r_i - \frac{\epsilon_i}{2}, r_i + \frac{\epsilon_i}{2}\right), \\ \gamma(t) & \text{otherwise.} \end{cases} \tag{9.12}$$

It is not smooth on all of $[0,1] \times [0,1]$ but only on the open subset $(0,1] \times [0,1]$. For every $\xi \in (0,1]$ the map

$$\gamma_\xi \colon [0,1] \to M, \quad t \mapsto c(\xi, t) \tag{9.13}$$

is a smooth path. It is time to recall the ordinary differential equation (8.69) from theorem 8.8 that characterizes the parallel transport along the smooth paths γ_ξ, where $\xi \in (0,1]$ is now treated as a parameter. Recall that for every $1 \le i \le n$ we have the uniform convergence

$$\lim_{\delta \to 0} \tilde{\gamma}_{i,\delta} = x_{(i)} \circ \gamma \tag{9.14}$$

as functions from $(r_i - \epsilon_i/2, r_i + \epsilon_i/2)$ to $\mathbb{R}^{\mathrm{Dim}(M)}$. This fact together with the smooth dependence on the parameter $\xi \in (0,1]$ leads us to the conclusion that for every $a \in E_p$ we have the convergence

$$\mathbb{P}_{\gamma_0}(a) = \lim_{\xi \to 0} \mathbb{P}_{\gamma_\xi}(a) \tag{9.15}$$

in E_q. This concludes the proof.

Definition 9.3. Let $\pi \colon E \to M$ be a vector bundle equipped with a covariant derivative operator ∇.
For every pair of vector fields $X, Y \in \Gamma(TM)$, we can define a $(1,1)$-tensor field $R(X,Y) \in \Gamma(E^{(1,1)})$, called the **curvature** of ∇, characterized by its action on a section $Z \in \Gamma(E)$ and a section $\omega \in \Gamma(E^*)$ of the dual bundle:

$$R(X,Y)(\omega, Z) = \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right). \tag{9.16}$$

Lemma 9.3. *The curvature $R(X,Y)$ is indeed a $(1,1)$-tensor field over $\pi \colon E \to M$.*

Proof 9.3. Recall that for every section $Z \in \Gamma(E)$ and every section $\omega \in \Gamma(E^*)$ of the dual bundle we have that

$$R(X,Y)(\omega, Z) = \omega\left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z\right) \tag{9.17}$$

is a smooth function. We can also immediately read off that $R(X,Y)$ is $C^\infty(M)$-linear in the first argument $\omega \in \Gamma(E^*)$. It is left to show that $R(X,Y)$ is also $C^\infty(M)$-linear in its second argument $Z \in \Gamma(E)$. This needs a more elaborate calculation. First, suppose that $W, Z \in \Gamma(E)$, then:

$$\begin{aligned} R(X,Y)(\omega, W + Z) &= \omega\left(\nabla_X \nabla_Y (W + Z) - \nabla_Y \nabla_X (W + Z) - \nabla_{[X,Y]}(W + Z)\right), && (9.18) \\ &= \omega\left(\nabla_X\left(\nabla_Y W + \nabla_Y Z\right) - \nabla_Y\left(\nabla_X W + \nabla_X Z\right) - \nabla_{[X,Y]} W - \nabla_{[X,Y]} Z\right), && (9.19) \\ &= \omega\left(\nabla_X \nabla_Y W + \nabla_X \nabla_Y Z - \nabla_Y \nabla_X W - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} W - \nabla_{[X,Y]} Z\right), && (9.20) \\ &= R(X,Y)(\omega, W) + R(X,Y)(\omega, Z), && (9.21) \end{aligned}$$

where we used item 3 from definition 8.1 two consecutive times. Second, suppose that $Z \in \Gamma(E)$ and $\varphi \in C^\infty(M)$. Then:

$$\begin{aligned}
R(X,Y)(\omega,\varphi\cdot Z) &= \omega\left(\nabla_X\nabla_Y(\varphi Z) - \nabla_Y\nabla_X(\varphi Z) - \nabla_{[X,Y]}(\varphi Z)\right), &(9.22)\\
&= \omega\left(\nabla_X\left(Y(\varphi)\cdot Z + \varphi\cdot\nabla_Y(Z)\right) - \nabla_Y\left(X(\varphi)\cdot Z + \varphi\cdot\nabla_X(Z)\right) - [X,Y](\varphi)\cdot Z\right.\\
&\quad \left.-\varphi\cdot\nabla_{[X,Y]}(Z)\right), &(9.23)\\
&= \omega\left(X(Y(\varphi))\cdot Z + Y(\varphi)\cdot\nabla_X(Z) + X(\varphi)\cdot\nabla_Y(Z) + \varphi\cdot\nabla_X\nabla_Y Z\right.\\
&\quad -Y(X(\varphi))\cdot Z - X(\varphi)\cdot\nabla_Y(Z) - Y(\varphi)\cdot\nabla_X(Z) - \varphi\cdot\nabla_Y\nabla_X Z\\
&\quad \left.-[X,Y](\varphi)\cdot Z - \varphi\cdot\nabla_{[X,Y]}(Z)\right), &(9.24)\\
&= \omega\left(\varphi\cdot\nabla_X\nabla_Y Z - \varphi\cdot\nabla_Y\nabla_X Z - \varphi\cdot\nabla_{[X,Y]}(Z)\right), &(9.25)\\
&= \varphi\cdot R(X,Y)(\omega,Z), &(9.26)
\end{aligned}$$

where we used item 4 from definition 8.1 two consecutive times. Due to proposition 6.1, this proves that $R(X,Y) \in \Gamma(E^{(1,1)})$ is a $(1,1)$-tensor field over $\pi\colon E \to M$.

Remark 9.2. Even though we have not made the appropriate extension of proposition 6.1 rigorous in details, it should be stated that we can go beyond lemma 9.3 and identify $R \in \Gamma(TM^{(0,2)} \otimes E^{(1,1)})$ as a section of the tensor product bundle $TM^{(0,2)} \otimes E^{(1,1)}$ of the $(0,2)$-tensor product bundle over $\pi_{TM}\colon TM \to M$ and the $(1,1)$-tensor product bundle over $\pi\colon E \to M$. All we have to do is prove that $R(X,Y)(\omega,Z)$ is $C^\infty(M)$-linear in the arguments $X \in \Gamma(TM)$ and $Y \in \Gamma(TM)$.

Definition 9.4. A covariant derivative operator ∇ on a vector bundle $\pi\colon E \to M$ is said to be **curvature-free** if for every two vector fields $X, Y \in \Gamma(TM)$, the tensor field $R(X,Y) \in \Gamma(E^{(1,1)})$ vanishes.

Theorem 9.4. *Let $\pi\colon E \to M$ be a vector bundle of rank k equipped with a covariant derivative operator ∇. If the restricted holonomy group $\mathrm{Hol}_0^\nabla(p)$ of ∇ is trivial at every point $p \in M$, then ∇ is curvature-free.*

Proof 9.4. Suppose $p \in M$ is a point. We will show that there exists a neighbourhood U of p on which $R(X,Y)$ vanishes for all $X, Y \in \Gamma(TU)$.

There exists a chart $x \in \mathcal{A}_M$ at p with the property that $x(p) = 0$ and $x[\mathrm{Dom}(x)] = \mathbb{R}^{\mathrm{Dim}(M)}$, thus satisfying the hypothesis of lemma 8.4, which states that

$$\Psi^{(x)}\colon\ \mathrm{T}\,\mathrm{Dom}(x) \to \mathrm{Dom}(x), \quad X \mapsto x^{-1}\left(x(\pi_{TM}(X)) + x_*(X)\right) \tag{9.27}$$

is a smooth map. For every $X \in T_pM$, define the smooth path

$$\gamma_X^{(x)}\colon [0,1] \to M, \quad t \mapsto \Psi^{(x)}(tX). \tag{9.28}$$

Furthermore, we define the map

$$\chi\colon\ \mathrm{Dom}(x) \to T_pM, \quad q \mapsto x^i(q)\left(\frac{\partial}{\partial x^i}\right)_p \tag{9.29}$$

For every point $q \in \mathrm{Dom}(x)$, the smooth path $\gamma_{\chi(q)}$ agrees with $[0,1] \to M, t \mapsto x^{-1}(tx(q))$. Suppose now that we are given a basis $\tilde{b}_1, \dots, \tilde{b}_n \in E_p$ of the fibre E_p over p of $\pi\colon E \to M$. For $1 \leq i \leq k$, we can define the map

$$b_i\colon\ \mathrm{Dom}(x) \to E, \quad q \mapsto \mathbb{P}_{\gamma^{(x)}_{\chi(q)}}\left(\tilde{b}_i\right), \tag{9.30}$$

which is smooth due to the fact that $\Psi^{(x)}\colon T\,\mathrm{Dom}(x) \to \mathrm{Dom}(x)$ satisfies the hypothesis of item 5 from definition 8.7. This proves that $b_i\colon \mathrm{Dom}(x) \to E$ is a local section of $\pi\colon E \to M$ over $\mathrm{Dom}(x)$. By definition, $b_i \circ \gamma_{\chi(q)}$ is parallelly transported along $\gamma_{\chi(q)}$. Furthermore, the collection $b_1(q), \dots, b_k(q) \in E_q$ is a basis of the fibre E_q over q due to item 1 from definition 8.7.

Let $q, q' \in \mathrm{Dom}(x)$ be two points and let $\delta\colon [0,1] \to \mathrm{Dom}(x)$ be a smooth path from q to q'. We claim that $b_i(q') = \mathbb{P}_\delta(b_i(q))$. First, note that the path concatenation $(\gamma_{X(q)} * \delta) * \overline{\gamma}_{X(q')}$ is a contractible piecewise smooth loop based at p. By the hypothesis that the restricted holonomy group $\mathrm{Hol}_0^\nabla(p)$ of ∇ at p is trivial, we conclude that

$$\mathrm{id}_{E_p} = \mathbb{P}_{(\gamma_{X(q)}*\delta)*\overline{\gamma}_{X(q')}} = \mathbb{P}_{\overline{\gamma}_{X(q')}} \circ \mathbb{P}_\delta \circ \mathbb{P}_{\gamma_{X(q)}}, \tag{9.31}$$

where we made use of item 3 from definition 8.7. Using also item 2 from definition 8.7 then yields

$$\mathbb{P}_{\gamma_{X(q')}} = \mathbb{P}_\delta \circ \mathbb{P}_{\gamma_{X(q)}}. \tag{9.32}$$

Applying this equation to $\tilde{b}_i \in E_p$ proves the claim. Since this holds for an arbitrary smooth path $\delta\colon [0,1] \to M$ from an arbitrary point $q \in \mathrm{Dom}(x)$ to an arbitrary point $q' \in \mathrm{Dom}(x)$, this proves that $b_i \circ \delta$ is parallelly transported along every curve $\delta\colon (\tau_1, \tau_2) \to \mathrm{Dom}(x)$. This concludes the proof that $b_1, \ldots, b_k \in \Gamma(E|_{\mathrm{Dom}(x)})$ is a parallelly transported local frame of $\pi\colon E \to M$. Theorem 8.9 now establishes that $\nabla_X b_i = 0$ for any local vector field $X \in \Gamma(T\,\mathrm{Dom}(x))$.

Finally, let $X, Y \in \Gamma(T\,\mathrm{Dom}(x))$ be local vector fields and let $s \in \Gamma(E|_{\mathrm{Dom}(x)})$ be a local section. On $\mathrm{Dom}(x)$, we can write

$$s = s^i_{(b)} \cdot b_i. \tag{9.33}$$

A quick calculation yields

$$\begin{aligned} R(X,Y)s &= \nabla_X \nabla_Y s - \nabla_Y \nabla_X s - \nabla_{[X,Y]} s, && (9.34)\\ &= \nabla_X \left(Y\left(s^i_{(b)}\right) \cdot b_i\right) - \nabla_Y \left(X\left(s^i_{(b)}\right) \cdot b_i\right) - [X,Y]\left(s^i_{(b)}\right) \cdot b_i, && (9.35)\\ &= X\left(Y\left(s^i_{(b)}\right)\right) \cdot b_i - Y\left(X\left(s^i_{(b)}\right)\right) \cdot b_i - [X,Y]\left(s^i_{(b)}\right) \cdot b_i && (9.36)\\ &= 0, && (9.37) \end{aligned}$$

where we have applied item 4 from definition 8.1 multiple times, combined with the facts that $\nabla_X b_i = 0$, $\nabla_Y b_i = 0$ and $\nabla_{[X,Y]} b_i = 0$. This concludes the proof.

Theorem 9.5. *Let $\pi\colon E \to M$ be a vector bundle of rank k equipped with a covariant derivative operator ∇. If ∇ is curvature-free, then its restricted holonomy group $\mathrm{Hol}_0^\nabla(p)$ at a point $p \in M$ is trivial.*

Proof 9.5. We only need to show that for any smooth contractible loop based at p $\gamma\colon [0,1] \to M$ the parallel transport $\mathbb{P}_\gamma(a)$ of any point $a \in E_p$ along γ coincides with a. The same then holds for a piecewise smooth loop based at p by lemma 9.2.

Suppose $\gamma\colon [0,1] \to M$ is a smooth contractible loop based at p. By definition, there exists a path homotopy $\tilde{c}\colon [0,1] \times [0,1] \to M$ from γ to the constant loop $[0,1] \to M, t \mapsto p$. Both γ and the constant loop $[0,1] \to M, t \mapsto p$ are smooth. Ultimately due to the Whitney embedding theorem it is always possible[10] to find a smooth path homotopy $c\colon [0,1] \times [0,1] \to M$ from γ to the constant loop $[0,1] \to M, t \mapsto p$.

Let $\tilde{b}_1, \ldots, \tilde{b}_k \in E_p$ be a basis of E_p. We can define a global frame of the pullback bundle c^*E. For every $1 \leq i \leq k$, define

$$b_i\colon [0,1] \times [0,1] \to E, \quad (r,t) \mapsto P_{\gamma_r, \tilde{b}_i}(t), \tag{9.38}$$

where $\gamma_r\colon [0,1] \to M, t \mapsto c(r,t)$. Each b_i is smooth due to the fact that $[0,1]$ is compact and due to the smooth dependence of solutions to ordinary differential equations on its parameters, as discussed earlier already.

Likewise, suppose $a \in E_p$ and define the smooth section along c given by

$$s\colon [0,1] \times [0,1] \to E, \quad (r,t) \mapsto P_{\gamma_r, a}(t). \tag{9.39}$$

Our goal is to prove that $\mathbb{P}_\gamma(a) = s(0,1) = a$.

[10] *Introduction to Smooth Manifolds* (2nd edition) by J. M. Lee, Page 142, Theorem 6.29

First, note that by definition of s we implemented $\nabla^c_{\partial_{\mathrm{id}^2}} s = 0$, so we have for the curvature that

$$\left(\nabla^c_{\partial_{\mathrm{id}^2}} \nabla^c_{\partial_{\mathrm{id}^1}} s\right)(r,t) = \underbrace{R\left(c_*(\partial_{\mathrm{id}^2})_{r,t}, c_*(\partial_{\mathrm{id}^1})_{r,t}\right)}_{=0} s(r,t). \tag{9.40}$$

However, by hypothesis, the right hand side vanishes. Expressing $\nabla^c_{\partial_{\mathrm{id}^1}} s$ in terms of the frame $b_1, \ldots, b_k$ as

$$\left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)(r,t) = \left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)^i_{(b)}(r,t) \cdot b_i(r,t) \tag{9.41}$$

and using the fact that for each $1 \leq i \leq k$ it holds that $\nabla^c_{\partial_{\mathrm{id}^2}} b_i = 0$, we can rewrite (9.40) as

$$\partial_2 \left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)^i_{(b)}(r,t) \cdot b_i(r,t) = 0, \tag{9.42}$$

providing an ordinary differential equation for each $1 \leq i \leq k$:

$$\partial_2 \left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)^i_{(b)}(r,t) = 0. \tag{9.43}$$

We conclude that

$$\left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)^i_{(b)}(r,1) = \left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)^i_{(b)}(r,0). \tag{9.44}$$

By definition of s, however, the right hand side vanishes, leading us to the equation

$$\left(\nabla^c_{\partial_{\mathrm{id}^1}} s\right)(r,1) = 0. \tag{9.45}$$

Define the smooth loop

$$\delta\colon [0,1] \to M, \quad r \mapsto c(r,1). \tag{9.46}$$

As a matter of fact, δ is the constant loop based at p. So the map

$$\hat{b}_i\colon [0,1] \to E, \quad r \mapsto \tilde{b}_i \tag{9.47}$$

is a global frame of the pullback bundle $\delta^* E$. Defining also the section along δ

$$\hat{s}\colon [0,1] \to E, \quad r \mapsto s(r,1), \tag{9.48}$$

equation (9.45) states precisely that $\hat{s}$ is parallelly transported along δ

$$\nabla^\delta_{\partial_{\mathrm{id}^1}} \hat{s} = 0. \tag{9.49}$$

Using the fact that $\hat{b}_i$ are parallelly transported along δ as well, this yields

$$\partial_1 \hat{s}^i_{(\hat{b})} = 0 \tag{9.50}$$

But then $\hat{s}^i_{\hat{b}}(1) = \hat{s}^i_{\hat{b}}(0)$. And so $\hat{s}(1) = \hat{s}(0)$. Thus, by definition, $s(1,1) = s(0,1)$. But $\gamma_1\colon [0,1] \to M, t \mapsto c(1,t)$ is the constant path. By lemma 8.3, it holds $s(1,1) = s(1,0) = a$. Thus $s(0,1) = a$, the desired result.

Remark 9.3. Theorem 9.4 and theorem 9.5 establish a light version of the Ambrose-Singer theorem.[AS53]

10 The Global Geometry of Teleparallel Gravity

10.1 The relation to spin structures

In physics, a *spacetime* is usually defined as a four-dimensional connected smooth manifold equipped with a Lorentzian bundle metric on the tangent bundle $\pi_{TM}: TM \to M$. Via the definition of a topological manifold, we usually require a spacetime to be in particular Hausdorff and paracompact. For spacetimes, the Hausdorff property implements distinguishability of different events. The paracompactness property is not an assumption but rather follows from the existence of the Lorentzian metric g on $\pi_{TM}: TM \to M$. An accessible proof of this result can be found in the Appendix of [Ger68].

Definition 10.1. A **spacetime** is a four-dimensional, connected, smooth manifold whose tangent bundle is equipped with a Lorentzian bundle metric $g \in \Gamma(TM^{(0,2)})$.

A *spacetime* can have further topological properties, such as compactness, simple-connectedness, orientability, time-orientability, space-orientability, or different causality conditions, such as stable causality or global hyperbolicity. Some of which we might want to assume or rule out.

For an in-depth discussion about the manifold topology of spacetimes refer to [GH79]. We will summarize here some of the most important results. As a first step, we are interested in particular in orientability, time-orientability and simply-connectedness.

Indeed, there are physical arguments that might justify to forbid spacetimes that are non-orientable from a physical point of view. This does not apply to non-time-orientable though.

The argument sometimes brought forward in order to motivate the prohibition of non-time-orientable spacetimes goes as follows: First, any physical observer perceives a time-orientation. And second, there should be agreement between the different observers. On the first glance this argument appears convincing. However, it is not clear how two observers with opposing time-orientations would even come to the conclusion that their time-orientations do not agree. How could the two observers even communicate with each other, respecting their respective time-orientations? One might falsely interpret the impossibility of such communication as an indication in favour of time-orientability. There is also no inconsistency arising from the laws of fundamental physics. For, if the first observer emits a particle then the second observer simply would interpret it as an anti-particle, due to the CPT invariance of quantum field theory. It is also important to remember that entropy is an observer-dependent concept, so there is no intrinsic way to determine the time-direction of another observer or composed system. In conclusion, the above is not a proper justification of why to forbid non-time-orientable spacetimes.

At least, there is some experimental support as to why to forbid non-orientable spacetimes. Suppose two observers with agreeing time-orientations meet at an event A and separate. Later, at an event B, the first observer sends a neutrino towards the second observer. The second observer detects the particle at an event C. If the spacetime that harbours the observers were non-orientable, the chirality of the neutrino could invert, from left-handed to right-handed chirality. Up to now, no right-handed neutrinos have been observed. Consequently, as we accept the absence of right-handed neutrinos as a fundamental law of nature, the universe cannot be non-orientable. For a more careful argument refer to pages 43 to 48 of Geroch's PhD thesis [Ger67].

We might prefer not to make any assumptions based on elementary particle physics. For this case, however, we have a handy mathematical trick at hand that enables us to anyway assume some of these properties without any harm done to the physical intactness of the theory.

Definition 10.2. Let $(M, \mathcal{O}_M, \mathcal{A}_M)$ be a connected smooth manifold.

A **covering manifold** $\tilde{M}$ of M is the the total space of a fibre bundle $\pi_{\text{cov}} : \tilde{M} \to M$ with fibre F a 0-dimensional (smooth) manifold.

Remark 10.1. It is true that every connected smooth manifold M admits a simply-connected covering manifold $\tilde{M}$. In fact, the latter is unique up to diffeomorphism and is called the **universal covering manifold**. Even more is true.

On the one hand, if M comes endowed with the structure of a Lorentzian metric on its tangent bundle $\pi_{TM} : TM \to M$, then there exists a unique Lorentzian metric on the tangent bundle $\pi_{T\tilde{M}} : T\tilde{M} \to \tilde{M}$ of the universal covering manifold $\tilde{M}$ such that the smooth covering map $\pi_{\text{cov}} : \tilde{M} \to M$ is a local isometry. (Indeed, we can simply pull back the $(0,2)$-tensor g along π_{cov}.)

On the other hand, if M is a connected Lie group G, then the universal covering manifold $\tilde{G}$ admits a unique Lie group structure that makes the smooth covering map $\pi_{\text{cov}} : \tilde{G} \to G$ into a group homomorphism. We say that $\tilde{G}$ is the **universal covering group**.

The idea is, whenever confronted with a spacetime M that is not simply-connected, to consider instead its simply-connected universal covering manifold $\tilde{M}$ endowed with the pullback Lorentzian bundle metric $\pi_{\text{cov}}{}^*g \in \Gamma(T\tilde{M}^{(0,2)})$ along the smooth covering map $\pi_{\text{cov}} : \tilde{M} \to M$. This idea is backed up by the argument that the spacetimes M and $\tilde{M}$ cannot be distinguished by any local experiment.

In this manner, we can always find a simply-connected spacetime that serves us. It is important to point out that this is merely a change of perspective, enabled by the fact that experiments can only be carried out locally. It is a handy change of perspective nevertheless, at least from the mathematical point of view: Together with the simply-connectedness of $\tilde{M}$ we also get orientability, time-orientability and space-orientability for free.

The orientability and time-orientability of $\tilde{M}$ come in particularly handy. For it enables us to reduce the structure group of the orthogonal frame bundle $\hat{\pi} : \mathrm{Fr}_g(TM) \to M$ further from the pseudo-orthogonal group $\mathrm{O}(1,3)$ to the restricted Lorentz group $\mathrm{SO}^+(1,3)$. The latter is the connected component of the pseudo-orthogonal group $\mathrm{O}(1,3)$ that contains the identity and encompasses all proper orthochronous Lorentz transformations. Interestingly, the restricted Lorentz group $\mathrm{SO}^+(1,3)$ is not simply-connected. The importance of this property cannot be understated. If it were not for this, all of quantum field theory would work in a very different way. Quantum field theory heavily uses the smooth covering homomorphism

$$\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3) \tag{10.1}$$

from the universal covering group $\mathrm{SL}(2,\mathbb{C})$ to the restricted Lorentz group $\mathrm{SO}^+(1,3)$. Physicists often like to obfuscate this fact by treating representations of $\mathrm{SL}(2,\mathbb{C})$ as if they were representations of $\mathrm{SO}^+(1,3)$. This is only partially true. It is true that a representation of $\mathrm{SL}(2,\mathbb{C})$ always gives rise to a *projective representation* of $\mathrm{SO}^+(1,3)$ and only sometimes gives rise to a representation of $\mathrm{SO}^+(1,3)$. Indeed, we call a field an integer-spin field if it transforms according to a representation of $\mathrm{SL}(2,\mathbb{C})$ that descends to a representation of $\mathrm{SO}^+(1,3)$, and otherwise, a half-integer-spin field. Let us make this idea rigorous before moving on, with a slight generalization.

Suppose for a moment we were to allow different spacetime dimensions. The orthogonal frame bundle of the a spacetime of dimension $k \geq 3$ would have as its fibre the pseudo-orthogonal group $\mathrm{O}(1,k-1)$. There exists a direct generalization of the proper orthochronous Lorentz transformations that form the connected component $\mathrm{SO}^+(1,k-1)$ of $\mathrm{O}(1,k-1)$ that contains the identity element. $\mathrm{SO}^+(1,k-1)$ is connected and, as it turns out, not simply-connected with non-trivial fundamental group[11] given by either $\mathbb{Z}$ (for $k = 3$) or

[11] See *An Introduction to Geometrical Physics* (2nd edition) by R. Aldrovandi and J. G. Pereira, Chapter 3, Section 3.2.2

$\mathbb{Z}_2$ (for $k \geq 4$).[12] This leads us to conclude[13] that for every $k \geq 3$ the universal covering homomorphism $\rho_k\colon \operatorname{Spin}(1,k-1) \to \mathrm{SO}^+(1,k-1)$ is a double cover. The universal covering group of $\mathrm{SO}^+(1,k-1)$ is usually referred to as the $(1,k-1)$**-spin group** $\operatorname{Spin}(1,k-1)$. Thus the definition:

Definition 10.3. Let $\pi\colon E \to M$ be an orientable vector bundle of rank k equipped with a time-orientable Lorentzian bundle metric $g \in \Gamma(E^{(0,2)})$.

A **spin structure** for g is a ρ_k-principal bundle homomorphism along id_M from a $\operatorname{Spin}(1,k)$-principal bundle

$$\pi_{\operatorname{Spin}(1,k-1)}\colon \operatorname{SpinFr}_g(E) \to M \tag{10.2}$$

over M to the structure group reduction

$$\pi_{\mathrm{SO}^+(1,k-1)}\colon \operatorname{Fr}_g(E)|_{\mathrm{SO}^+(1,k-1)} \to M \tag{10.3}$$

of the orthogonal frame bundle to the connected component $\mathrm{SO}^+(1,k-1)$ of the pseudo-orthogonal group $\mathrm{O}(1,k-1)$ that contains the identity element. Here, $\rho_k\colon \operatorname{Spin}(1,k-1) \to \mathrm{SO}^+(1,k-1)$ is the universal covering homomorphism of $\mathrm{SO}^+(1,k-1)$.

The $\operatorname{Spin}(1,k-1)$-principal bundle $\pi_{\operatorname{Spin}(1,k-1)}\colon \operatorname{SpinFr}(E) \to M$ is then called a **spin frame bundle** for $\pi\colon E \to M$ with respect to g.

Remark 10.2. A vector bundle $\pi\colon E \to M$ of rank k equipped with a Lorentzian bundle metric $g \in \Gamma(E^{(0,2)})$ may fail to admit a spin structure or may admit multiple inequivalent spin structures. Here, inequivalence is to be understood in the sense that there may exist spin frame bundles $\rho\colon P \to M$ and $\rho'\colon P' \to M$ for $\pi\colon E \to M$ between which there does not exist a $\mathrm{id}_{\operatorname{Spin}(1,k-1)}$-principal bundle isomorphism along id_M.

Returning to the context of 4-dimensional spacetimes, we have the following definition, in full analogy to definition 10.3.

Definition 10.4. A **spin structure** for an orientable and time-orientable spacetime (M,g) is a ρ-principal bundle homomorphism along id_M from a $\mathrm{SL}(2,\mathbb{C})$-principal bundle

$$\pi_{\mathrm{SL}(2,\mathbb{C})}\colon \operatorname{SpinFr}_g(TM) \to M \tag{10.4}$$

over M to the structure group reduction

$$\pi_{\mathrm{SO}^+(1,3)}\colon \operatorname{Fr}_g(TM)|_{\mathrm{SO}^+(1,3)} \to M \tag{10.5}$$

of the orthogonal frame bundle to the restricted Lorentz group $\mathrm{SO}^+(1,3)$, where $\rho\colon \mathrm{SL}(2,\mathbb{C}) \to \mathrm{SO}^+(1,3)$ is the universal covering homomorphism of $\mathrm{SO}^+(1,3)$.[14]

The $\mathrm{SL}(2,\mathbb{C})$-principal bundle $\pi_{\mathrm{SL}(2,\mathbb{C})}\colon \operatorname{SpinFr}(TM) \to M$ is then called a **spin frame bundle** for (M,g).

Remark 10.3. In full analogy to remark 10.2, it is true that a spacetime (M,g) may fail to admit a spin structure or may admit multiple inequivalent spin structures. Much effort was investigated in the past to study under which conditions a spacetime admits a spin structure.

Indeed, let us recall, without proof, the main theorem from [Ger68]:

Theorem 10.1. *A non-compact spacetime admits a spin structure if and only if it admits a global orthonormal frame.*

---

[12] See *Lie Groups, Lie Algebras, and Representations* (2nd edition) by B. Hall, Chapter 13, Proposition 13.10 in combination with Section 13.3

[13] *Supersymmetry for Mathematicians: An Introduction* by V. S. Varadarajan, Chapter 5, Page 200, Theorem 5.4.7

[14] Compare with the definitions found in [Mil63], [Lic68] and [Pen68].

For the proof of theorem 10.1 refer to [Ger68]. An immediate corollary following from the combination of theorem 10.1 with theorem 9.4 is the following:

Corollary 10.2. *A non-compact spacetime that admits a spin structure also admits a curvature-free metric-compatible covariant derivative operator* ∇.

In [HS79], considered one of the first works on teleparallel gravity, a Weitzenböck connection is characterized as a curvature-free metric-compatible covariant derivative operator on the tangent bundle $\pi_{TM}: TM \to M$ of the spacetime (M, g). Corollary 10.2 hence tells us that every non-compact spacetime (M, g) with spin structure admits a description in terms of teleparallel gravity.

This leaves us with the question as to whether our universe is non-compact.

As it turns out, there are arguments for why compact spacetimes are rarely considered, based on assumptions about the causal structure of a spacetime. Indeed, a *stably causal*[15] spacetime cannot be compact.[16] See also the discussion in [HE73]. On the other hand, *stable causality* by itself is another matter of discussion: A *stably-causal* spacetime is necessarily time-orientable. However, in the paragraph that discussed the time-orientability we concluded that we cannot rule out the possibility of non-time-orientability altogether. Although we have seen that the universal covering of a spacetime is time-orientable, it does not seem particularly useful to discuss the causality of the universal covering of a spacetime instead of the causality of a spacetime itself. There is, however, once more a handy mathematical trick: It turns out that the universal covering of a spacetime is non-compact.[17] Consequently, we have the following result:

Corollary 10.3. *If the universal covering* $(\tilde{M}, \tilde{g})$ *of a spacetime* (M, g) *admits a spin structure then it also admits a curvature-free metric-compatible covariant derivative operator* $\tilde{\nabla}$.

A question that remains is whether or not we can find a curvature-free metric-compatible covariant derivative operator on the spacetime (M, g) itself. The answer is in general no, since the induced global orthonormal frame $\tilde{e}_1, \ldots, \tilde{e}_{\mathrm{Dim}(M)}$ is not necessarily the lift of any global orthonormal frame $e_1, \ldots, e_{\mathrm{Dim}(M)}$.

In any case, if the spacetime in question is simply-connected, it is its own universal covering and we have the following simplified situation:

Corollary 10.4. *A simply-connected spacetime admits a spin structure if and only if it admits a curvature-free metric-compatible covariant derivative operator.*

Indeed, the inverse direction is due to the fact that a Weitzenböck connection defines an absolute parallelism if the spacetime is simply-connected, a direct consequence of theorem 9.5.

[15] This terminology is due to Stephen Hawking. See [Haw69].

[16] This result is due to E. H. Kronheimer and R. Penrose. See [KP67].

[17] It suffices to employ the following argument. Take the *Euler characteristic* of a manifold M given by

$$\chi(M) = \sum_{p=0}^{\mathrm{Dim}(M)} (-1)^{-p} \,\mathrm{Dim}\left(H^p(M)\right)$$

as defined by [MS74] in terms of the dimensions of the *cohomology groups* of M. If M is compact and orientable, *Poincaré duality* (Theorem 11.10 from [MS74]) tells us that for every $0 \le p \le \mathrm{Dim}(M)$ we have the equivalence

$$H^p(M) \cong H^{\mathrm{Dim}(M)-p}(M).$$

In turn, *de Rham duality* (Theorem 18.14 from [Lee13]) enables us to substitute the cohomology group $H^p(M)$ by the *de Rham cohomology group* $H^p_{\mathrm{dR}}(M)$. The Euler characteristic of any four-dimensional compact orientable smooth manifold can thus be calculated according to

$$\chi(M) = 2\,\mathrm{Dim}\left(H^0_{\mathrm{dR}}(M)\right) - 2\,\mathrm{Dim}\left(H^1_{\mathrm{dR}}(M)\right) + \mathrm{Dim}\left(H^2_{\mathrm{dR}}(M)\right).$$

Now, a compact connected simply-connected smooth manifold M has $\mathrm{Dim}\left(H^0_{\mathrm{dR}}(M)\right) = 1$ (M has one connected component) and $\mathrm{Dim}\left(H^1_{\mathrm{dR}}(M)\right) = 0$ (M is simply-connected). Consequently, M has Euler characteristic $\chi(M) \ge 2$. Suppose the universal covering $(\tilde{M}, \tilde{g})$ of a spacetime (M, g) were compact. Then $\chi(\tilde{M}) \ge 2$. However, since $\tilde{M}$ is simply-connected it is true that $(\tilde{M}, \tilde{g})$ is time-orientable. There exists a no-where vanishing vector field on $\tilde{M}$. The *Poincaré-Hopf theorem* [Hop27] implies that $\chi(\tilde{M}) = 0$. A contradiction. This proves that the universal covering of a spacetime is non-compact.

10.2 From general relativity to teleparallel gravity

Until now we have not mentioned the *torsion* of a covariant derivative operator at all. We postponed its definition intentionally this far in order to highlight the fact that torsion is a property that is exclusive to covariant derivative operators that are defined on a vector bundle $\pi\colon E \to M$ that is vector-bundle-isomorphic to the tangent bundle $\pi_{TM}\colon TM \to M$ of the base manifold M:

Definition 10.5. Let M be a smooth manifold whose tangent bundle $\pi_{TM}\colon TM \to M$ is equipped with a covariant derivative operator ∇. The **torsion** of ∇ is the $(1,2)$-tensor field

$$T\colon \Gamma(T^*M)\times\Gamma(TM)\times\Gamma(TM)\to C^\infty(M), \quad (\omega,X,Y)\mapsto \omega\left(\nabla_X Y - \nabla_Y X - [X,Y]\right). \tag{10.6}$$

Observe that torsion can only be defined for covariant derivative operators defined on tangent bundles.

Traditionally speaking, the only fame of the torsion tensor is due to the fact that it is usually required to vanish. This the case of General Relativity but not exclusively. The field of Riemannian geometry also usually employs the unique (and always existing) torsion-free metric-compatible covariant derivative operator, also known as *Levi-Civita* covariant derivative operator.

The original subject of teleparallel gravity is to choose an alternative metric-compatible covariant derivative operator: one that is curvature-free. As discussed in the preceding section, differently from the unique torsion-free metric-compatible covariant derivative operator that always exists, a curvature-free metric-compatible covariant derivative operator may fail to exist. This is the case for all non-parallelizable spacetimes but not exclusively.

An example is provided once more by the maximally extended Schwarzschild spacetime (M,g). It is parallelizable and simply-connected as its underlying manifold is homeomorphic to $S^2\times\mathbb{R}^2$. Suppose there exists a curvature-free metric-compatible covariant derivative operator ∇. Since M is simply-connected, the holonomy group $\mathrm{Hol}^\nabla(p)$ is trivial for any point $p\in M$. We thus can construct a global orthonormal frame using the parallel transport system of ∇. However, no such global orthonormal frame can exist for the maximally extended Schwarzschild spacetime. A contradiction. This proves by contradiction that there does not exist a curvature-free metric-compatible covariant derivative operator for the maximally extended Schwarzschild spacetime. (The geodesically-incomplete parts of the spacetime separated by the event horizon, referred to as interior and exterior, both independently admit curvature-free metric-compatible, for they independently admit global orthonormal frames.)

10.3 Ungeometrizing gravity

We suppose we are given a spacetime (M,g) together with a curvature-free metric-compatible covariant derivative operator ∇. On any simply-connected patch $U\subseteq M$ we may construct a parallelly-transported orthonormal frame $h_1,\dots,h_{\mathrm{Dim}(M)}\in\Gamma(TU)$. We will need the following definition:

Definition 10.6. Let U be an open set of M. A frame $b_1,\dots,b_{\mathrm{Dim}(M)}\in\Gamma(TU)$ of TM over U is said to be **holonomic** if for every $1\le i,j\le \mathrm{Dim}(M)$ the Lie-bracket

$$[b_i,b_j]=0 \tag{10.7}$$

of b_i and b_j vanishes on all of U. By contrast, a frame that fails to be **holonomic** is said to be **anholonomic**.

If U is contractible, then a **holonomic** frame over U can be used to define coordinates on U by integration of the coframe $e^1,\dots,e^{\mathrm{Dim}(M)}$. This is part of the Poincaré lemma.[18]

Remark 10.4. Note that the notion of a holonomic frame has nothing to do with the holonomy group of a covariant derivative operator.

[18] The most general and complete exposition is likely found in *Differential Forms and Applications* by M. P. Do Carmo.

This becomes particularly clear when we consider a parallelly transported frame $e_1, \dots, e_{\mathrm{Dim}(M)} \in \Gamma(TU)$ with respect to a curvature-free covariant derivative operator ∇. The restricted holonomy group $\mathrm{Hol}_0^\nabla(p)$ vanishes but this means in no way that $e_1, \dots, e_{\mathrm{Dim}(M)}$ is a holonomic frame.

Definition 10.7. Let U be an open set of M and let $b_1, \dots, b_{\mathrm{Dim}(M)} \in \Gamma(TU)$ be a frame of TM over U. For every $1 \leq i, j \leq \mathrm{Dim}(M)$, we can express the Lie-bracket of b_i and b_j as

$$[b_i, b_j] = {\Xi_{(b)}}^k{}_{ij} \cdot b_k, \tag{10.8}$$

where the functions ${\Xi_{(b)}}^k{}_{ij} \in C^\infty(U)$ are called the **coefficients of anholonomy of** $b_1, \dots, b_{\mathrm{Dim}(M)}$.

On a small enough neighbourhood of any point $p \in M$ of spacetime it is possible to find a parallelly transported orthonormal frame $h_1, \dots, h_{\mathrm{Dim}(M)} \in \Gamma(TU)$ and a holonomic frame $b_1, \dots, b_{\mathrm{Dim}(M)} \in \Gamma(TU)$. For, if $x \in \mathcal{A}_M$ is a chart at p with simply-connected domain $\mathrm{Dom}(x)$, then

$$\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^{\mathrm{Dim}(M)}} \tag{10.9}$$

is a holonomic frame over $\mathrm{Dom}(M)$, and due to the fact that $\mathrm{Dom}(x)$ is simply-connected we can construct a parallelly-transported orthonormal frame $h_1, \dots, h_{\mathrm{Dim}(M)}$ over $\mathrm{Dom}(x)$.

We might ask whether it is possible to find such frames globally. We already established that if M is simply-connected, then a parallelly-transported orthonormal frame exists globally. There also are obstructions to the existence of a global holonomic frame. There even exist contractible four-dimensional smooth manifolds that do not admit a global frame.[19] If a spacetime admits a global chart $x \in \mathcal{A}_M$, then of course there exists a global holonomic frame.

Suppose we are given an open set $U \subseteq M$ together with a parallelly transported orthonormal frame $h_1, \dots, h_{\mathrm{Dom}(M)}$ over U, and a holonomic frame $b_1, \dots, b_{\mathrm{Dom}(M)}$ over U. We can express the orthonormal frame $h_1, \dots, h_{\mathrm{Dim}(M)}$ in terms of the holonomic frame $b_1, \dots, b_{\mathrm{Dim}(M)}$ and the coframe $h^1, \dots, h^{\mathrm{Dim}(M)}$ in terms of $b^1, \dots, b^{\mathrm{Dim}(M)}$:

$$h_i = {h_i}^{(b)j} b_j, \quad h^i = {h^i}_{(b)j} b^j. \tag{10.10}$$

Where it holds that

$$ {h^i}_{(b)k} {h_j}^{(b)k} = \delta^i_j, \tag{10.11}$$

by definition, and

$$g(h_i, h_j) = \eta_{ij}, \quad g(b_i, b_j) = {h^k}_{(b)i} {h^l}_{(b)j} \eta_{kl} \tag{10.12}$$

due to the fact that $h_1, \dots, h_{\mathrm{Dim}(M)}$ is orthonormal. With respect to the fixed holonomic frame $b_1, \dots, b_{\mathrm{Dim}(M)}$, the component functions ${h^i}_{(b)j}$ thus parametrize the Lorentzian bundle metric g locally. As such we can formulate the dynamic equations of general relativity locally in terms of the component functions ${h^i}_{(b)j}$ of the cotetrad instead. It seems that the degrees of freedom were increased from 10 to 16, while we continue with only 10 Einstein equations. Indeed, the component functions ${h^i}_{(b)j}$ are not uniquely determined by the Einstein equations. This is due to the fact that there exists a diverse family of component functions ${h^i}_{(b)j}$ that give rise to the same Lorentzian bundle metric g.

[19] Let N be a contractible four-dimensional smooth manifold that is not homeomorphic to $\mathbb{R}^4$. (Such manifolds do exist. See [Maz61] and [Koh21].) Suppose there exists a global holonomic frame $b_1, \dots, b_4$ on it with corresponding dual frame $b^1, \dots, b^4$. Since N is contractible, the Poincaré lemma from [Do 94] ensures that the closed 1-forms $b^1, \dots, b^4$ are exact, thus providing a global chart for N. However, N is not homeomorphic to $\mathbb{R}^4$. A contradiction.

In the case that the spacetime (M, g) is such that we can find a parallelly transported orthonormal frame $h_1, \ldots, h_{\mathrm{Dim}(M)}$ and a holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)}$ globally, we can employ the following interesting interpretation: We can regard the global holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)}$ as an parallelly transported orthonormal frame to a hypothetical gravity-free metric $\tilde{g}$. Locally, this frame provides Cartesian coordinates for the gravity-free metric $\tilde{g}$.

In the case of the Schwarzschild spacetime we might regard the as background truth the holonomic frame given by:

$$
\begin{aligned}
b_0 &= \frac{\partial}{\partial t}, \\
b_1 &= \sin(\theta)\cos(\phi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial}{\partial \theta} - \frac{1}{r\sin(\theta)}\sin(\phi)\frac{\partial}{\partial \phi}, \\
b_2 &= \sin(\theta)\sin(\phi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial}{\partial \theta} + \frac{1}{r\sin(\theta)}\cos(\phi)\frac{\partial}{\partial \phi}, \\
b_3 &= \cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta},
\end{aligned}
\tag{10.13}
$$

where (t, r, θ, ϕ) are the Schwarzschild coordinates for the Schwarzschild metric

$$
g = \left(1 - \frac{r_S}{r}\right) \mathrm{d}t \otimes \mathrm{d}t - \frac{1}{1 - \frac{r_S}{r}} \mathrm{d}r \otimes \mathrm{d}r - r^2 \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin(\theta)^2 \,\mathrm{d}\phi \otimes \mathrm{d}\phi\right), \tag{10.14}
$$

while being merely ordinary spherical coordinates for the gravity-free metric given by

$$
\tilde{g} = b^0 \otimes b^0 - \sum_{i=1}^{3} b^i \otimes b^i = \mathrm{d}t \otimes \mathrm{d}t - \mathrm{d}r \otimes \mathrm{d}r - r^2 \left(\mathrm{d}\theta \otimes \mathrm{d}\theta + \sin(\theta)^2 \,\mathrm{d}\phi \otimes \mathrm{d}\phi\right). \tag{10.15}
$$

A global orthonormal frame for the Schwarzschild metric is given by:

$$
\begin{aligned}
h_0 &= \frac{1}{\sqrt{1 - \frac{r_S}{r}}}\frac{\partial}{\partial t}, \\
h_1 &= \sqrt{1 - \frac{r_S}{r}}\sin(\theta)\cos(\phi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\cos(\phi)\frac{\partial}{\partial \theta} - \frac{1}{r\sin(\theta)}\sin(\phi)\frac{\partial}{\partial \phi}, \\
h_2 &= \sqrt{1 - \frac{r_S}{r}}\sin(\theta)\sin(\phi)\frac{\partial}{\partial r} + \frac{1}{r}\cos(\theta)\sin(\phi)\frac{\partial}{\partial \theta} + \frac{1}{r\sin(\theta)}\cos(\phi)\frac{\partial}{\partial \phi}, \\
h_3 &= \sqrt{1 - \frac{r_S}{r}}\cos(\theta)\frac{\partial}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial}{\partial \theta}.
\end{aligned}
\tag{10.16}
$$

The component functions ${h_i}^{(b)j}$ are in this case given by:

$$
{h_i}^{(b)j} = \begin{pmatrix}
\frac{1}{\sqrt{1-\frac{r_S}{r}}} & 0 & 0 & 0 \\
0 & 1 + f(r)\sin(\theta)^2\cos(\phi)^2 & f(r)\sin(\theta)^2\sin(\phi)\cos(\phi) & f(r)\sin(\theta)\cos(\theta)\cos(\phi) \\
0 & f(r)\sin(\theta)^2\sin(\phi)\cos(\phi) & 1 + f(r)\sin(\theta)^2\sin(\phi)^2 & f(r)\sin(\theta)\cos(\theta)\sin(\phi) \\
0 & f(r)\sin(\theta)\cos(\theta)\cos(\phi) & f(r)\sin(\theta)\cos(\theta)\sin(\phi) & 1 + f(r)\cos(\theta)^2
\end{pmatrix}, \tag{10.17}
$$

where

$$
f(r) = \sqrt{1 - \frac{r_S}{r}} - 1. \tag{10.18}
$$

We observe that in the case of the Schwarzschild metric, the holonomic frame $b_0, \ldots, b_3$ can be continuously (but not uniformly) transformed into the orthonormal frame $h_0, \ldots, h_3$: It suffices to gradually increase the Schwarzschild radius $r_S = \frac{GM}{c^2}$, starting from 0 corresponding to the gravity-free case. With respect to the reference frame $b_0, \ldots, b_3$ we thus found a way to switch on gravity gradually.

The worldlines of massive objects on which no other force acts continues to be given by the auto-parallelly transported curves with respect to the *Levi-Civita* covariant derivative operator $\mathbb{V}$ on TM:

$$\mathbb{V}^{\gamma}_{\partial_{\mathrm{id}}}\dot{\gamma} = 0. \tag{10.19}$$

We would like to alternatively use the curvature-free metric-compatible covariant derivative ∇ defined by the global orthonormal frame $h_0, \dots, h_3$ through

$$\forall 0 \leq i, j \leq 3\colon \quad \nabla_{h_i} h_j = 0. \tag{10.20}$$

In proposition 8.1 we have seen that there exists a tensor field $K \in \Gamma(TM^{(1,2)})$ that satisfies

$$\nabla_X Y - \mathbb{V}_X Y = K(X, Y). \tag{10.21}$$

Using this relation, we may rewrite the equation of motion (10.19) of massive objects in the following way:

$$\nabla^{\gamma}_{\partial_{\mathrm{id}}}\dot{\gamma} = K(\dot{\gamma}, \dot{\gamma}). \tag{10.22}$$

The tensor field K is called the **contorsion tensor**. We may express this equation in terms of the holonomic frame $b_0, \dots, b_3$.

To this end, let us first consider a more general setting. Let M be a smooth manifold with a Lorentzian bundle metric g on the tangent bundle, a metric-compatible covariant derivative operator ∇, and a local frame $e_1, \dots, e_{\mathrm{Dim}(M)} \in \Gamma(TU)$. To get a rough overview, let us write down the components of the torsion tensor T and the curvature tensor R of ∇. We have

$$\begin{aligned} T(e^c, e_a, e_b) &= e^c\left(\nabla_{e_a} e_b - \nabla_{e_b} e_a - [e_a, e_b]\right) \\ &= {\Gamma_{(e)}}^c{}_{ba} - {\Gamma_{(e)}}^c{}_{ab} - {\Xi_{(e)}}^c{}_{ab} \end{aligned} \tag{10.23}$$

for the torsion tensor and

$$\begin{aligned} R(e^d, e_a, e_b, e_c) &= e^d\left(\nabla_{e_a}\nabla_{e_b} e_c - \nabla_{e_b}\nabla_{e_a} e_c - \nabla_{[e_a, e_b]} e_c\right) \\ &= e^d\left(\nabla_{e_a}\left({\Gamma_{(e)}}^m{}_{cb} e_m\right) - \nabla_{e_b}\left({\Gamma_{(e)}}^m{}_{ca} e_m\right) - {\Xi_{(e)}}^m{}_{ab}{\Gamma_{(e)}}^n{}_{cm} e_n\right) \\ &= e_a\left({\Gamma_{(e)}}^d{}_{cb}\right) - e_b\left({\Gamma_{(e)}}^d{}_{ca}\right) + {\Gamma_{(e)}}^d{}_{ma}{\Gamma_{(e)}}^m{}_{cb} - {\Gamma_{(e)}}^d{}_{mb}{\Gamma_{(e)}}^m{}_{ca} - {\Xi_{(e)}}^m{}_{ab}{\Gamma_{(e)}}^d{}_{cm} \end{aligned} \tag{10.24}$$

for the curvature tensor. The metric-compatibility condition reads

$$0 = e_c\left({g_{(e)}}_{ab}\right) - {\Gamma_{(e)}}^d{}_{ac}\,{g_{(e)}}_{db} - {\Gamma_{(e)}}^d{}_{bc}\,{g_{(e)}}_{ad}. \tag{10.25}$$

Now suppose that the local frame $e_1, \dots, e_{\mathrm{Dim}(M)}$ is orthonormal and suppose that ∇ is metric-compatible. Then (10.25) yields

$${\Gamma_{(e)}}^{ba}{}_c = -{\Gamma_{(e)}}^{ab}{}_c. \tag{10.26}$$

We can use this equation in order to arrive at a handy expression for the coefficient functions ${\Gamma_{(e)}}^c{}_{ab}$:

$${\Gamma_{(e)}}^c{}_{ab} = \frac{1}{2}\left({T_{(e)}}_a{}^c{}_b + {T_{(e)}}_b{}^c{}_a - {T_{(e)}}^c{}_{ab} + {\Xi_{(e)}}_a{}^c{}_b + {\Xi_{(e)}}_b{}^c{}_a - {\Xi_{(e)}}^c{}_{ab}\right). \tag{10.27}$$

This expression holds in particular for the *Levi-Civita* covariant derivative operator $\mathbb{V}$ as well as for a curvature-free metric-compatible covariant derivative operator ∇, if existent. Suppose from now onwards that ∇ is a curvature-free metric-compatible covariant derivative operator and that $h_1, \dots, h_{\mathrm{Dim}(M)} \in \Gamma(TU)$ is a local parallelly transported orthonormal frame. The components of the **contorsion tensor** with respect to this frame then read

$${K_{(h)}}^c{}_{ab} = -\frac{1}{2}\left({\Xi_{(h)}}_a{}^c{}_b + {\Xi_{(h)}}_b{}^c{}_a - {\Xi_{(h)}}^c{}_{ab}\right). \tag{10.28}$$

Given a local holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)} \in \Gamma(TU)$, the coefficients of anholonomy of $h_1, \ldots, h_{\mathrm{Dim}(M)}$ may be expressed in terms of the component functions ${h_i}^{(b)^j}$:

$$\Xi_{(h)}{}^c{}_{ab} = \left({h_a}^{(b)^k}\, {h_b}^{(b)^l} - {h_b}^{(b)^k}\, {h_a}^{(b)^l}\right) b_l \left({h^c}_{(b)_k}\right) . \tag{10.29}$$

Equation (10.28) in terms of the component functions ${h_i}^{(b)^j}$ is

$$\begin{aligned} K_{(h)}{}^c{}_{ab} = \frac{1}{2} \Big(&\left({h_a}^{(b)^k}\, {h_b}^{(b)^l} - {h_b}^{(b)^k}\, {h_a}^{(b)^l}\right) b_l \left({h^c}_{(b)_k}\right) \\ &- \eta_{ai}\, \eta^{cj} \left({h_j}^{(b)^k}\, {h_b}^{(b)^l} - {h_b}^{(b)^k}\, {h_j}^{(b)^l}\right) b_l \left({h^i}_{(b)_k}\right) \\ &- \eta_{bi}\, \eta^{cj} \left({h_j}^{(b)^k}\, {h_a}^{(b)^l} - {h_a}^{(b)^k}\, {h_j}^{(b)^l}\right) b_l \left({h^i}_{(b)_k}\right) \Big) . \end{aligned} \tag{10.30}$$

Its components with respect to the holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)}$ are given by:

$$\begin{aligned} K_{(b)}{}^l{}_{mn} &= {h_c}^{(b)^l}\, {h^a}_{(b)_m}\, {h^b}_{(b)_n}\, K_{(h)}{}^c{}_{ab} \\ &= \frac{1}{2} \Big(b_n \left({h^c}_{(b)_m}\right) {h_c}^{(b)^l} - b_m \left({h^c}_{(b)_n}\right) {h_c}^{(b)^l} \\ &\quad - {h^a}_{(b)_m}\, {h_c}^{(b)^l}\, {h_j}^{(b)^k}\, \eta_{ai}\, \eta^{cj} \left(b_n \left({h^i}_{(b)_k}\right) - b_k \left({h^i}_{(b)_n}\right)\right) \\ &\quad - {h^b}_{(b)_n}\, {h_c}^{(b)^l}\, {h_j}^{(b)^k}\, \eta_{bi}\, \eta^{cj} \left(b_m \left({h^i}_{(b)_k}\right) - b_k \left({h^i}_{(b)_m}\right)\right) \Big) . \end{aligned} \tag{10.31}$$

We can finally write down the equation of motion (10.19) in terms of the component functions ${h_i}^{(b)^j}$ alone:

$$\begin{aligned} &\dot\gamma^m_{(b)}\, b_m \left(\dot\gamma^l_{(b)}\right) b_l + \dot\gamma^m_{(b)}\, \dot\gamma^n_{(b)}\, b_m \left({h^k}_{(b)_n}\right) {h_k}^{(b)^l}\, b_l \\ &\qquad = -\dot\gamma^m_{(b)}\, \dot\gamma^n_{(b)}\, {h^b}_{(b)_n}\, {h_c}^{(b)^l}\, {h_j}^{(b)^k}\, \eta_{bi}\, \eta^{cj} \left(b_m \left({h^i}_{(b)_k}\right) - b_k \left({h^i}_{(b)_m}\right)\right) b_l . \end{aligned} \tag{10.32}$$

Back to the case of the Schwarzschild metric with global orthonormal frame $h_0, \ldots, h_3$ and global holonomic frame $b_0, \ldots, b_3$, we may plug in the component functions ${h_i}^{b^j}$ from equation (10.17) in order to obtain the equation of motion in terms of the Schwarzschild coordinates t, r, θ and ϕ. Note that in this case the equation that we obtain is the equation that characterizes auto-parallel curves with respect to the *Levi-Civita* covariant derivative operator, expressed in the global chart[20] x defined by the holonomic frame $b_0, \ldots, b_3$. Indeed, the second term on the left hand side of (10.32) together with the right hand side reproduce the value of the coefficient functions of the *Levi-Civita* covariant derivative operator with respect to the global chart x, expressed in terms of the component functions ${h_i}^{(b)^j}$.

10.4 Translation-group flavoured teleparallelism

So far so good about traditional teleparallel gravity. The second research line in teleparallel gravity is to try to describe gravity as a gauge theory, in this case of the translation group $\mathbb{R}^4$. The beginning of this era is roughly set by "Einstein Lagrangian as the translational Yang-Mills Lagrangian" by Y. M. Cho.

In the first part of this work we have seen that every principal bundle $\pi_{\mathrm{T}} \colon P \to M$ of the translation group $\mathbb{R}^4$ over the spacetime M is trivial. If one works within a framework where one wishes to establish an affine bundle isomorphism

$$\Phi \colon TM \to P \tag{10.33}$$

[20] Even though the Schwarzschild spacetime is not contractible, there exists such a global chart. In fact, we defined the holonomic frame $b_0, \ldots, b_3$ in terms of the Schwarzschild coordinates t, r, θ and ϕ precisely in such a way that b^i coincides with the differential $\mathrm{d}x^i$ of the Cartesian coordinate function x^i, with respect to the gravity-free metric $\tilde{g}$.

between the tangent bundle $\pi_{TM}: TM \to M$ and the principal bundle $\pi_{\mathrm{T}}: P \to M$ of the translation group $\mathbb{R}^4$, then it is implied that the tangent bundle $\pi_{TM}: TM \to M$ is trivial as well. In other words, M needs to be parallelizable.

In chapter 9, we had seen that a curvature-free metric-compatible covariant derivative operator ∇ on a simply-connected spacetime provides a global orthonormal frame that parallelizes the tangent bundle.

In the previous section, we had discussed theorem 10.1 from which it follows that a non-compact spacetime M with spin-structure admits a global orthonormal frame and hence is in particular parallelizable.

In this light, we conclude that the existence of an affine bundle isomorphism between the principal bundle $\pi_{\mathrm{T}}: P \to M$ of the translation group $\mathbb{R}^4$ and the tangent bundle is no additional obstruction.

Time to dive deeper into translation-group flavoured teleparallel gravity. The framework of translation-group flavoured teleparallel gravity is the following: Suppose we are given a smooth manifold M together with a global holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)} \in \Gamma(TM)$. There exist various different Lorentzian bundle metrics g on the tangent bundle $\pi_{TM}: TM \to M$. As we have seen, for many of them, but not necessarily all of them, there exist a global orthonormal frame $h_1, \ldots, h_{\mathrm{Dim}(M)}$. Consider only those Lorentzian bundle metrics g that do admit a global orthonormal frame. We may then express the global frame $h_1, \ldots, h_{\mathrm{Dim}(M)}$ in terms of the global holonomic frame and the coframe $h^1, \ldots, h^{\mathrm{Dim}(M)}$ in terms of $b^1, \ldots, b^{\mathrm{Dim}(M)}$:

$$h_i = {h_i}^{(b)^j} b_j, \quad h^i = {h^i}_{(b)_j} b^j. \tag{10.34}$$

Where it holds that

$$ {h^i}_{(b)_k} {h_j}^{(b)^k} = \delta^i_j, \tag{10.35}$$

by definition, and

$$g(h_i, h_j) = \eta_{ij}, \quad g(b_i, b_j) = {h^k}_{(b)_i} {h^l}_{(b)_j} \eta_{kl} \tag{10.36}$$

due to the fact that $h_1, \ldots, h_{\mathrm{Dim}(M)}$ is orthonormal. With respect to the fixed global holonomic frame, the component functions ${h^i}_{(b)_j}$ thus parametrize the Lorentzian bundle metric g. As such we can formulate the dynamic equations of general relativity in terms of the component functions ${h^i}_{(b)_j}$ of the cotetrad instead. It seems that the degrees of freedom were increased from 10 to 16, while we continue with only 10 Einstein equations. Indeed, the component functions ${h^i}_{(b)_j}$ are not uniquely determined by the Einstein equations. This is due to the fact that there exists a diverse family of component functions ${h^i}_{(b)_j}$ that give rise to the same Lorentzian bundle metric g.

We may decompose the component functions ${h_i}^{(b)^j}$ in the following way:

$$ {h_i}^{(b)^j} = \delta^j_i + {B_i}^{(b)^j}. \tag{10.37}$$

Note that the δ^j_i happen to coincide with the component functions ${b_i}^{(b)^j}$ of the holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)}$ expressed with respect to itself. The idea of the functions ${B_i}^{(b)^j}$ is thus to describe only the difference between the global frame $h_1, \ldots, h_{\mathrm{Dim}(M)}$ and the global holonomic frame $b_1, \ldots, b_{\mathrm{Dim}(M)}$. In fact, for each $1 \leq i \leq 4$ we have the global vector field

$$B_i = {B_i}^{(b)^j} b_j. \tag{10.38}$$

Observe that the collection $B_1, \ldots, B_{\mathrm{Dim}(M)}$ is not required to define a frame.

At this point, it may look like a rather artificial construction to express the component functions ${h_i}^{(b)^j}$ in terms of ${B_i}^{(b)^j}$. Now, however, suppose that we are given a principal bundle $\pi: P \to M$ of the translation group $\mathbb{R}^4$ over M together with a principal bundle connection, a $\mathrm{Lie}(\mathbb{R}^4)$-valued 1-form on P

$$\omega = \sum_{i=1}^{4} \omega^i \, \xi_i^*, \tag{10.39}$$

where ω^i are just ordinary 1-forms on P and ξ_i^* are the generators of the Lie group $\mathbb{R}^4$. If $\pi\colon E \to M$ is a vector bundle associated to $\pi\colon P \to M$, then a covariant derivative on $\pi\colon E \to M$ is induced. In components this looks like

$$(\mathbb{\nabla}_{b_i}\psi) = b_i(\psi^k)e_k + \beta_i{}^j \xi_j^*(\psi), \tag{10.40}$$

where $\beta_i{}^j$ are the components of the Yang-Mills field and $e_1, \ldots, e_k$ is some local frame of the vector bundle $\pi\colon E \to M$. Now, teleparallel gravity uses the covariant derivative operator $\mathbb{\nabla}$ to parametrize the frame $h_1, \ldots, h_{\mathrm{Dim}(M)}$, using the gauge potentials, which are identified with the component functions:

$$B_i{}^{(b)j} = \beta_i{}^j. \tag{10.41}$$

At his point, the generators ξ_j^* are often forcefully identified with the derivations b_j. (10.40) becomes

$$(\mathbb{\nabla}_{b_i}\psi) = b_i(\psi^k)e_k + \beta_i{}^j b_j(\psi^k)e_k, \tag{10.42}$$

which for $\beta_i{}^j = B_i{}^{(b)j}$ yields

$$(\mathbb{\nabla}_{b_i}\psi) = b_i(\psi^k)e_k + B_i{}^{(b)j} b_j(\psi^k)e_k = h_i(\psi^k)e_k. \tag{10.43}$$

This makes it seem like we can interpret the derivations h_i as a covariant derivative obtained from the derivations b_i, with $B_i{}^{(b)j}$ the translational Yang-Mills field with respect to the local frame e_k. This, however, is misleading: First of all, a vector bundle cannot be associated to a principal bundle of the translation group $\mathbb{R}^4$ in first place. And second, even if $\pi\colon E \to M$ were associated to the principal bundle $\pi\colon P \to M$ of the translation group $\mathbb{R}^4$, then the identification of the generators ξ_j^* with the derivations b_i would be erroneous, for the action of the generators ξ_j^* on ψ^k is solely determined by the Lie group left action of $\mathbb{R}^4$ on the typical fibre of $\pi\colon E \to M$. It cannot yield something that depends on the neighbouring fibres, but the derivation b_i does depend on the neighbouring fibres and thus cannot be a suitable candidate.

In order to see that, let us suppose we are given a principal bundle $\hat{\pi}\colon P \to M$ over a spacetime M together with an associated affine bundle $\pi\colon E \to M$. Suppose $\omega \in \Omega(P, T_0(\mathbb{R}^4))$ is a $\mathbb{R}^4$-valued 1-form on P, defining a connection on the principal bundle $\hat{\pi}\colon P \to M$. Given a local (or global) section $\sigma\colon U \to P$ of $\hat{\pi}\colon P \to M$, we can pull back the $\mathbb{R}^4$-valued 1-form ω onto the open subset U of M, obtaining a $\mathbb{R}^4$-valued 1-form $\sigma^*\omega$ defined locally on $U \subseteq M$.

$$\beta \in \Omega(M, T_0(\mathbb{R}^4)) \tag{10.44}$$

is a $T_0(\mathbb{R}^4) \cong \mathbb{R}^4$-valued form on M, the Yang-Mills field.

What we can do is consider an affine bundle $\pi\colon E \to M$ associated to the principal bundle $\pi\colon P \to M$ of the translation group $\mathbb{R}^4$. At this point we run into multiple problems.

- $\pi\colon E \to M$ is an affine bundle. It is not clear what the correct generalization of a covariant derivative operator on an affine bundle should be. A straightforward generalization of the vector bundle case does not exist since there are no appropriate versions of items 1 to 4. See also remark 8.1
- All the matter types known to date are represented by sections of vector bundles, may it be tensor fields over the tangent bundle or sections of vector bundles associated to the spin frame bundle. Since none of these vector bundles can be canonically associated to the principal bundle $\hat{\pi}\colon P \to M$ of the translation group $\mathbb{R}^4$, the conclusion is that none of the fields living in them couple to the Yang-Mills fields of the principal connection on $\hat{\pi}\colon P \to M$.

11 Conclusions

The geometric description of teleparallel gravity has sparked controversies between mathematicians [FHL19] and physicists [PO19]. Aldrovandi and Pereira developed in [AP13] an alternative formalism for gravity, starting from a Weitzenböck connection[21] [HS79] towards a description in terms of the translation group T_4. The mathematicians criticize in [FHL19] that in order to equip the tangent bundle of a smooth manifold with the structure of a principal bundle with respect to the action of the translation group, it is necessary that the tangent bundle is trivial.

We confirmed this result from [FHL19] in theorem 4.3. Indeed, a spacetime whose frame bundle is non-trivial cannot accommodate a translation-group structure. But how big of an obstruction is this for the applicability of translation-group flavoured teleparallel gravity? We can answer this question in two complementary ways.

On the one hand, we can give an answer that is motivated by the primitive teleparallel gravity from [HS79] itself. We have seen that a curvature-free metric-compatible covariant derivative on a simply-connected spacetime provides a global orthonormal frame. But then the frame bundle of the underlying spacetime is trivial. So in this case, there is no obstruction for equipping the tangent bundle with an action of the translation group. The question that is left is which action of the translation group to choose and with which utility.

On the other hand, we may give a physical argument as to why there is no unreasonable obstruction. As physicists, when we are given a spacetime, we are not directly interested in its orthogonal frame bundle but instead we are interested in its spin frame bundle, if existent. This is because not every spacetime allows spinor fields to be defined on it. Most physicists would agree that any sensible spacetime should allow spinor fields to be definable. Geroch proved that a non-compact four-dimensional spacetime admits a spin-structure if and only if there exists a global orthonormal frame [Ger68]. In this light, any physically sensible non-compact spacetime admits a description in the framework of teleparallel gravity. In fact, a simply-connected (and non-compact[22]) spacetime admits a spin structure if and only if there exists a curvature-free metric-compatible covariant derivative for it. Nonetheless, there do exist spacetimes which do not admit a spin structure nor a description in terms of teleparallel gravity. A rather prominent example is the maximally extended Schwarzschild spacetime.

There are two important and interrelated questions left for future work to answer. First, how exactly do we construct the translation-group structure for a given spacetime? And second, does this translation-group structure properly define a principal bundle of the translation group?

At the present moment we do not have answers to any of the two questions, but we believe that the basis set in this work paves part of the path towards a rigorous description of translation-group flavoured teleparallel gravity.

[21] In our dictionary, this corresponds to what we defined as a curvature-free metric-compatible covariant derivative operator.
[22] See footnote 17 on page 74.

A Point-set topology

Definition A.1 (Topological space)**.** Let X be a set. A collection of subsets $\mathcal{O}_X \subseteq \mathcal{P}(X)$ is said to be a **topology** for the set X if $\mathcal{O}_X$ satisfies the following three following conditions:

(Top.1) Both $\emptyset$ and X are elements of $\mathcal{O}_X$. $\quad \emptyset, X \in \mathcal{O}_X$

(Top.2) Closure with respect to arbitrary unions. $\quad \forall \mathcal{U} \subseteq \mathcal{O}_X : \bigcup \mathcal{U} \in \mathcal{O}_X$

(Top.3) Closure with respect to finite intersections. $\quad \forall U_1, U_2 \in \mathcal{O}_X : U_1 \cap U_2 \in \mathcal{O}_X$

The tuple $(X, \mathcal{O}_X)$ is then said to compose a **topological space**. It is convention to refer to the elements of $\mathcal{O}_X$ as the **open subsets** of X. An open subset $U \in \mathcal{O}_X$ that contains a point $x \in X$ is called a **neighbourhood of** x.

Remark A.1. Every point $x \in X$ has a neighbourhood.

Definition A.2 (Topological basis)**.** Let X be a set. A **topological basis** for X is a collection $\mathcal{B}_X \subseteq \mathcal{P}(X)$ of subsets of X with such that

1. each point $x \in X$ is contained in at least one element $B \in \mathcal{B}$, and
2. for every two elements $A, B \in \mathcal{B}_X$ and every point $x \in A \cap B$ there exists an element $C \in \mathcal{B}_X$ such that $x \in C \subseteq A \cap B$.

Proposition A.1. *Let X be a set and $\mathcal{B}_X$ a topological basis for X. The set*

$$\mathcal{O}(\mathcal{B}_X) := \{A \in \mathcal{P}(X) \mid \forall x \in A \colon \exists B \in \mathcal{B}_X : x \in B \subseteq A\} \tag{A.1}$$

is a topology for X, the ***topology generated by*** *$\mathcal{B}_X$.*

Proof A.1. Items (1) and (2) from definition A.1 are readily verified. Item (3) holds true as well. For, if $U, V \in \mathcal{O}(\mathcal{B}_X)$ and $x \in U \cap V$, by definition of $\mathcal{O}(\mathcal{B}_X)$, there exist basis elements $A, B \in \mathcal{B}_X$ such that $x \in A \subseteq U$ and $x \in B \subseteq V$, respectively. Since $\mathcal{B}_X$ is a topological basis, there exists an element $C \in \mathcal{B}_X$ such that $x \in C \subseteq A \cap B$. But then $x \in C \subseteq U \cap V$. Hence $U \cap V \in \mathcal{O}(\mathcal{B}_X)$.

Proposition A.2. *Let $(X, \mathcal{O}_X)$ be a topological space and $\mathcal{B}_X \subseteq \mathcal{O}_X$ a collection of open subsets of X with the property that for every open set $U \in \mathcal{O}_X$ and every point $x \in U$ there exists an element $B \in \mathcal{B}_X$ satisfying $x \in B \subseteq U$. Then $\mathcal{B}_X$ is a topological basis for $\mathcal{O}_X$. Moreover, $\mathcal{O}_X$ is the topology $\mathcal{O}(\mathcal{B}_X)$ generated by $\mathcal{B}_X$.*

Proof A.2. First, let us prove item (1) of definition A.2. Let $x \in X$. By item (1) of definition A.1, we have that $X \in \mathcal{O}_X$. By hypothesis, there exists an element $B \in \mathcal{B}_X$ such that $x \in B \subseteq X$. Now, let us prove item (2) of definition A.2. To this end, suppose $A, B \in \mathcal{B}_X$ and $x \in A \cap B$. By hypothesis, $A, B \in \mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology, we have $A \cap B \in \mathcal{O}_X$. Hence there exists an element $C \in \mathcal{B}_X$ such that $x \in C \subseteq A \cap B$.

Let $U \in \mathcal{O}_X$. By hypothesis, for every point $x \in U$ there exists an element $B \in \mathcal{B}_X$ such that $x \in B \subseteq U$. Then, $U \in \mathcal{O}(\mathcal{B}_X)$, by definition of $\mathcal{O}(\mathcal{B}_X)$. Now, let $U \in \mathcal{O}(\mathcal{B}_X)$. Choose for every $x \in U$ a basis element $B_x \in \mathcal{B}_X$ satisfying $x \in B_x \subseteq U$. Then U is the union $\{B_x \mid x \in U\}$ of elements of $\mathcal{O}_X$. Since $\mathcal{O}_X$ is a topology, it follows that $U \in \mathcal{O}_X$.

Remark A.2. Proposition A.1 and proposition A.2 prove that every topology $\mathcal{O}_X$ for a set X is generated by some topological basis $\mathcal{B}_X$ for X.

Definition A.3 (Topological subbasis)**.** Let X be a set. A collection $\mathcal{S}_X \subseteq \mathcal{P}(X)$ of subsets of X is said to be a **topological subbasis** for X if every point $x \in X$ is contained in some $S \in \mathcal{S}_X$.

Proposition A.3. *Let X be a set and $\mathcal{S}_X$ a topological subbasis. The collection $\mathcal{B}(\mathcal{S}_X)$ of finitie intersections of elements of $\mathcal{S}_X$ is a topological basis for X.*

Proof A.3. Let $x \in X$. Then there exists $S \in \mathcal{S}_X$ such that $x \in S$. However, $S \in \mathcal{B}(\mathcal{S}_X)$, being a finite intersection of elements of $\mathcal{S}_X$. This proves item (1) of definition A.2. In order to prove item (2), it suffices to note that any two elements $A, B \in \mathcal{B}(\mathcal{S}_X)$ are finite intersections of elements of $\mathcal{S}_X$. Then $A \cap B$ is a finite intersection of elements of $\mathcal{S}_X$ and as such is an element of $\mathcal{B}(\mathcal{S}_X)$.

Definition A.4 (Second countable space)**.** A topological space $(X, \mathcal{O}_X)$ is said to be **second countable** if there exists a countable topological basis $\mathcal{B}_X$ for X that generates $\mathcal{O}_X$.

Definition A.5 (Hausdorff space)**.** A topological space $(X, \mathcal{O}_X)$ is said to be **Hausdorff** if for every two distinct points $x, y \in X$ there exist two disjoint open subsets $X, Y \in \mathcal{O}_X$ such that $x \in X$ and $y \in Y$, i.e.,

$$\forall x, y \in X \colon \; (x \neq y \implies \exists U, V \in \mathcal{O}_X \colon \; (x \in U \wedge y \in V \wedge U \cap V = \emptyset))\,. \tag{A.2}$$

Definition A.6 (Open cover)**.** Let $(X, \mathcal{O}_X)$ be a topological space. A collection of open sets $\mathcal{U} \subseteq \mathcal{O}_X$ is called an **open cover** of the topological space $(X, \mathcal{O}_X)$ if every point $x \in X$ is contained in an element $U \in \mathcal{U}$, i.e.,

$$X \subseteq \bigcup \mathcal{U}. \tag{A.3}$$

Definition A.7 (Subcover of an open cover)**.** Let $(X, \mathcal{O}_X)$ be a topological space and $\mathcal{U}$ an open cover of $(X, \mathcal{O}_X)$. A subset $\mathcal{V} \subseteq \mathcal{U}$ is said to be a **subcover** of $\mathcal{U}$ if it also an open cover of $(X, \mathcal{O}_X)$.

Definition A.8 (Lindelöf space)**.** A topological space $(X, \mathcal{O}_X)$ is said to be **Lindelöf** if every open cover admits a countable subcover.

Lemma A.4. *A second countable space is Lindelöf.*

Proof A.4. Let $\mathcal{B}_X$ be a countable topological basis that generates $\mathcal{O}_X$ and let $\mathcal{U}$ be an open cover of X. Define

$$\mathcal{I} \coloneqq \{B \in \mathcal{B} \mid \exists U \in \mathcal{U} \colon B \subseteq K\}\,. \tag{A.4}$$

Note that for every $B \in \mathcal{I}$ there exists at least one $U \in \mathcal{U}$ such that $B \subseteq U$. The axiom of choice guarantees the existence of a choice function $f \colon \mathcal{I} \to \mathcal{U}$ with the property that $\forall B \in \mathcal{I} \colon B \subseteq f(B)$. The image $f[\mathcal{I}]$ is a (countable) subcover of $\mathcal{U}$. For, if x is a point in X, there exists an open set $U \in \mathcal{U}$ such that $x \in U$, since $\mathcal{U}$ is an open cover. In turn, there exists a basis element $B \in \mathcal{B}_X$ such that $x \in B \subseteq U$. But then $B \in \mathcal{I}$ and, consequently, $x \in f(B)$.

Definition A.9 (Compact space)**.** Let $(X, \mathcal{O}_X)$ be a topological space. We say that $(X, \mathcal{O}_X)$ is **compact** if every open cover admits a finite subcover, i.e.,

$$\forall \mathcal{U} \subseteq \mathcal{O}_X \colon \; \left(\bigcup \mathcal{U} \supseteq X \implies \exists \mathcal{V} \subseteq \mathcal{U} \colon \; \left(\bigcup \mathcal{V} \supseteq X \wedge |\mathcal{V}| \in \mathbb{N}\right)\right). \tag{A.5}$$

Definition A.10 (Refinement of an open cover)**.** Let $(X, \mathcal{O}_X)$ be a topological space and $\mathcal{U}$ be an open cover of $(X, \mathcal{O}_X)$. Another open cover $\mathcal{V}$ is said to be a **refinement** of $\mathcal{U}$ if every element $V \in \mathcal{V}$ is a subset of an element $U \in \mathcal{U}$, i.e.,

$$\forall V \in \mathcal{V} \colon \exists U \in \mathcal{U} \colon V \subseteq U. \tag{A.6}$$

Remark A.3. A subcover $\mathcal{V}$ of an open cover $\mathcal{U}$ is a refinement of $\mathcal{U}$.

Definition A.11 (Locally finite cover)**.** Let $(X, \mathcal{O}_X)$ be a topological space and $\mathcal{U}$ be an open cover of $(X, \mathcal{O}_X)$. We say that $\mathcal{U}$ is **locally finite** if for every $x \in X$ there exists a neighbourhood W of x that has non-empty intersection with only finitely many elements of the open cover $\mathcal{U}$, i.e.,

$$\forall x \in X \colon \exists W \in \mathcal{O}_X \colon \ (x \in W \wedge |\{U \in \mathcal{U} \mid U \cap W \neq \emptyset\}| \in \mathbb{N}) . \tag{A.7}$$

Definition A.12 (Paracompact space)**.** A topological space $(X, \mathcal{O}_X)$ is **paracompact** if every open cover has a locally finite refinement.

Proposition A.5. *A compact space is paracompact.*

Proof A.5. Let $\mathcal{U}$ be an open cover of a compact space $(X, \mathcal{O}_X)$. By hypothesis there exists a finite subcover $\mathcal{V} \subseteq \mathcal{U}$. Note that $\mathcal{V}$ is also a refinement of $\mathcal{U}$.

For any point $x \in X$ and any neighbourhood W of the point x, it holds true that W has non-empty intersection with only finitely many elements of $\mathcal{V}$, given that $\mathcal{V}$ is finite. We conclude that $\mathcal{V}$ is a locally finite refinement of $\mathcal{U}$. Hence, $(X, \mathcal{O}_X)$ is paracompact.

Definition A.13 (Locally Euclidean space)**.** A topological space $(X, \mathcal{O}_X)$ is said to be **locally Euclidean of dimension** $d \in \mathbb{N}$ if for every point $x \in X$, there exist a neighbourhood U of x, an open set $V \in \mathcal{O}_{\mathbb{R}^d}$, and a homeomorphism $\sigma \colon U \to V$, also called a **chart at** x.

Definition A.14 (Atlas)**.** Let $(X, \mathcal{O}_X)$ be a locally Euclidean space of dimension $d \in \mathbb{N}$. An atlas for $(X, \mathcal{O}_X)$ is a collection $\mathcal{A}$ of charts whose domains cover X.

Lemma A.6. *A locally Euclidean space $(X, \mathcal{O}_X)$ of dimension $d \in \mathbb{N}$ with a countable atlas $\mathcal{A}$ is second countable.*

Proof A.6. We make use of the fact that $\mathbb{R}^d$ equipped with its standard topology $\mathcal{O}_{\mathbb{R}^d}$ is second countable. Let $\mathcal{B}_{\mathbb{R}^d}$ be a countable basis that generates $\mathcal{O}_{\mathbb{R}^d}$, for instance the collection of open balls with rational radius r and position c

$$\mathcal{B}_{\mathbb{R}^d} \coloneqq \big\{ \{x \in \mathbb{R}^d \mid \|x - c\| < r\} \mid c \in \mathbb{Q}^d \wedge r \in \mathbb{Q} \big\} . \tag{A.8}$$

We now define the set

$$\mathcal{B}_X \coloneqq \big\{ \alpha^{-1}[O] \mid (O, \alpha) \in \mathcal{B}_{\mathbb{R}^d} \times \mathcal{A} \big\} . \tag{A.9}$$

It is countable as the image of the cartesian product of the countable sets $\mathcal{B}_{\mathbb{R}^d}$ and $\mathcal{A}$. It is straightforward to check that $\mathcal{B}_X$ is indeed a topological basis that generates $\mathcal{O}_X$.

Corollary A.7. *A locally Euclidean space is Lindelöf if and only if it is second countable.*

Proof A.7. Any locally Euclidean space $(X, \mathcal{O}_X)$ admits an atlas $\mathcal{A}$. If $(X, \mathcal{O}_X)$ is Lindelöf, then $\mathcal{A}$ can be reduced to a countable atlas $\mathcal{A}'$. Lemma A.6 then states that $(X, \mathcal{O}_X)$ is second countable. The converse direction was the subject of lemma A.4.

Definition A.15. Let $(X, \mathcal{O}_X)$ be a topological space.

A **path in** X is a continuous map $\gamma \colon [0, 1] \to M$. γ is also said to be a **path from** $\gamma(0)$ **to** $\gamma(1)$.

Definition A.16. Let $(X, \mathcal{O}_X)$ be a topological space and $p, q \in X$ points in X. Suppose $\gamma_0 \colon [0, 1] \to X$ and $\gamma_1 \colon [0, 1] \to X$ are paths from p to q.

A **path homotopy from** γ_0 **to** γ_1 is a continuous map $c \colon [0, 1] \times [0, 1] \to M$ that satisfies

$$\forall t \in [0, 1] \colon \qquad c(0, t) = \gamma_0(t) \qquad \text{and} \qquad c(1, t) = \gamma_1(t), \tag{A.10}$$

$$\forall \xi \in [0, 1] \colon \qquad c(\xi, 0) = p \qquad \text{and} \qquad c(\xi, 1) = q. \tag{A.11}$$

For every $\xi \in [0, 1]$, the map $\gamma_\xi \colon [0, 1] \to M, t \mapsto c(\xi, t)$ is a path from p to q.

Definition A.17. Let $(X, \mathcal{O}_X)$ be a topological space and $p \in X$ a point in X.

A path $\gamma\colon [0,1] \to X$ from p to p is said to be a **loop based at** p.

Definition A.18. Let $(X, \mathcal{O}_X)$ be a topological space and $p \in X$ a point in X.

A loop $\gamma\colon [0,1] \to X$ at p is said to be **contractible** if there exists a path homotopy $c\colon [0,1] \times [0,1] \to X$ from γ to the constant path $\delta\colon [0,1] \to X, t \mapsto p$.

Definition A.19. A topological space $(X, \mathcal{O}_X)$ is said to be **simply-connected** if every loop in X is contractible.

Definition A.20. A topological space $(X, \mathcal{O}_X)$ is said to be **contractible** if there exists a continuous function $H\colon [0,1] \times X \to X$ and a point $y \in X$ such that for every $x \in X$ it holds that $H(0,x) = x$ and $H(1,x) = y$.

B Algebra

Definition B.1. Let G be a set and $\bullet\colon G \times G \to G, (g, h) \mapsto g \bullet h$ a binary operation on G. We call $(G, \bullet)$ a **group** if $\bullet$ satisfies:

$(\mathrm{A}^\bullet)$	Associativity of $\bullet$ on G	$\forall a, b, c \in G\colon a \bullet (b \bullet c) = (a \bullet b) \bullet c$
$(\mathrm{N}^\bullet)$	Neutral element of $\bullet$ on G	$\exists e \in G\colon \forall a \in G\colon e \bullet a = a = a \bullet e$
$(\mathrm{I}^\bullet)$	Inverse wrt. $\bullet$ on G	$\forall a \in G\colon \exists a^{-1} \in G\colon a^{-1} \bullet a = e = a \bullet a^{-1}$

Definition B.2. A group $(G, \bullet)$ is said to be **abelian** (or **commutative**) if the group operation $\bullet$ is commutative, i.e. if:

$(\mathrm{C}^\bullet)$	Commutativity of $\bullet$ on G	$\forall g, h \in G\colon g \bullet h = h \bullet g$

Definition B.3. Let $(G, \bullet)$ and $(H, \blacklozenge)$ be two groups.

A **group homomorphism from** G **into** H is a map $\varphi\colon G \to H$ that satisfies for every pair of elements $g_1, g_2 \in G$ the equation

$$\varphi\,(g_1 \bullet g_2) = \varphi(g_1) \blacklozenge \varphi(g_2). \tag{B.1}$$

Definition B.4. Let R be a set and $+\colon R \times R \to R$ and $\cdot\colon R \times R \to R$ be two binary operations on R.

The structure $(R, +, \cdot)$ is said to be a **ring** with addition $+$ and multiplication $\cdot$ if $(R, +)$ is an abelian group with neutral element denoted by $0 \in R$ and if $+$ and $\cdot$ additionally satisfy:

$(\mathrm{A}^\cdot)$	Associativity of $\cdot$ on R	$\forall a, b, c \in G\colon a \cdot (b \cdot c) = (a \cdot b) \cdot c$
$(\mathrm{D}^{+\cdot})$	Left-distribution of $\cdot$ over $+$	$\forall a, b, c \in R\colon (a + b) \cdot c = a \cdot c + b \cdot c$
$(\mathrm{D}^{\cdot+})$	Right-distribution of $\cdot$ over $+$	$\forall a, b, c \in R\colon a \cdot (b + c) = a \cdot b + a \cdot c$

Definition B.5. A ring $(R, +, \cdot)$ is said to be **unital** if the multiplication $\cdot$ admits a multiplicative neutral element, also called the **unit element**:

$(\mathrm{N}^\cdot)$	Multiplicative neutral element 1 of $\cdot$ on R	$\exists 1 \in R\colon \forall a \in R\colon 1 \cdot a = a = a \cdot 1$

Definition B.6. A ring $(R, +, \cdot)$ is said to be **commutative** if the multiplication $\cdot$ is commutative:

$(\mathrm{C}^\cdot)$	Commutativity of $\cdot$ on R	$\forall a, b \in R\colon a \cdot b = b \cdot a$

Exercise B.1. Convince yourself that the smooth functions $(C^\infty(M), +, \cdot)$ on M form a unital commutative ring.

Definition B.7. A commutative unital ring $(K, +, \cdot)$ is said to be a **field** if:

$(\mathrm{I}^\cdot)$	Inverse wrt. $\cdot$ on $K \setminus \{0\}$	$\forall a \in K\colon \exists a^{-1} \in K\colon a^{-1} \cdot a = 1 = a \cdot a^{-1}$

Note that then $(K \setminus \{0\}, \cdot)$ is an abelian group, too.

Definition B.8. Let $(R, +, \cdot)$ be a ring, $(A, \oplus)$ an abelian group and $\boxdot\colon R \times A \to A$ a map. We call $(A, \oplus, \boxdot)$ a **module over the ring** $(R, +, \cdot)$ if the **scalar multiplication** $\boxdot$ satisfies:

$A^{\cdot\boxdot}$	Associativity between $\cdot$ and $\boxdot$	$\forall \lambda, \mu \in R \colon \forall a \in A \colon (\lambda \cdot \mu) \boxdot a = \lambda \boxdot (\mu \boxdot a)$
$D^{\boxdot\oplus}$	Distributivity between $\boxdot$ and $\oplus$	$\forall \lambda \in R \colon \forall a, b \in A \colon \lambda \boxdot (a \oplus b) = (\lambda \boxdot a) \oplus (\lambda \boxdot b)$
$D^{+\boxdot}$	Distributivity between $+$ and $\boxdot$	$\forall \lambda, \mu \in R \colon \forall a \in A \colon (\lambda + \mu) \boxdot a = (\lambda \boxdot a) \oplus (\mu \boxdot a)$

Definition B.9. Let $(A, \oplus, \boxdot)$ be a module over a unital commutative ring $(R, +, \cdot)$. $(A, \oplus, \boxdot)$ is said to be **unital** if it satisfies:

$U^{\cdot\boxdot}$	Unit element $1 \in R$ as identity of scalar multiplication $\boxdot$	$\forall a \in A \colon 1 \boxdot a = a$

Exercise B.2. Let $(R, +, \cdot)$ be a commutative ring.

Show that $(R, +, \cdot)$ is a module over the commutative ring $(R, +, \cdot)$.

Remark B.1. Observe that vector space is nothing but a unital module over a field.

Definition B.10. Let $(V, \oplus_V, \boxdot_V)$ and $(W, \oplus_W, \boxdot_W)$ be two modules over the same commutative ring $(R, +, \cdot)$. A map $L \colon V \to W$ is said to be **linear**, or whenever confusion is possible, **R-linear**, if $L \colon (V, \oplus_V) \to (W, \oplus_W)$ is a group homomorphism and satisfies

$$\forall v \in V \colon \forall \lambda \in R \colon \quad L\left(\lambda \underset{V}{\boxdot} v\right) = \lambda \underset{W}{\boxdot} L(v). \tag{B.2}$$

Definition B.11. Let $(V_1, \oplus_{V_1}, \boxdot_{V_1})$, ..., $(V_r, \oplus_{V_r}, \boxdot_{V_r})$ and $(W, \oplus_W, \boxdot_W)$ be modules over the same commutative ring $(R, +, \cdot)$. A map $L \colon V_1 \times \cdots \times V_r \to W$ is said to be **multilinear**, or whenever confusion is possible, **R-multilinear**, if for every $1 \leq s \leq r$ and every $v_1 \in V_1$, ..., $v_{s_1} \in V_{s_1}$, $v_{s+1} \in V_{s+1}$, ..., $v_r \in V_r$, the map

$$L(v_1, \ldots, v_{s-1}, \cdot, v_{s+1}, \ldots, v_r) \colon V_r \to W, \quad v_s \mapsto L(v_1, \ldots, v_{s-1}, v_s, v_{s+1}, \ldots, v_r) \tag{B.3}$$

is linear. The set of **R-multilinear maps** from $V_1 \times \cdots \times V_r$ to W is denoted by $\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)$.

Definition B.12. The **addition** of multilinear maps is the operation

$$\begin{aligned} \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r)(W)}{\oplus} \colon \mathrm{Mult}_R(V_1 \times \cdots \times V_r, W) \times \mathrm{Mult}_R(V_1, \times \cdots \times V_r, W) &\to \mathrm{Mult}_R(V_1 \times \cdots \times V_r, W), \\ (L, L') &\mapsto L \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)}{\oplus} L', \end{aligned} \tag{B.4}$$

where

$$L \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)}{\oplus} L' \colon V_1 \times \cdots \times V_r \to W, \quad (v_1, \ldots, v_r) \mapsto L(v_1, \ldots, v_r) \underset{W}{\oplus} L'(v_1, \ldots, v_r). \tag{B.5}$$

Definition B.13. The **scalar multiplication** of multilinear maps is the operation

$$\begin{aligned} \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r)(W)}{\boxdot} \colon R \times \mathrm{Mult}_R(V_1, \times \cdots \times V_r, W) &\to \mathrm{Mult}_R(V_1 \times \cdots \times V_r, W), \\ (\lambda, L) &\mapsto \lambda \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)}{\boxdot} L, \end{aligned} \tag{B.6}$$

where

$$\lambda \underset{\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)}{\boxdot} L \colon V_1 \times \cdots \times V_r \to W, \quad (v_1, \ldots, v_r) \mapsto \lambda \underset{W}{\boxdot} L(v_1, \ldots, v_r). \tag{B.7}$$

Exercise B.3. The set of multilinear maps $\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)$ together with its addition $\oplus_{\mathrm{Mult}_R(V_1, \times \cdots \times V_r, W)}$ and its scalar multiplication $\boxdot_{\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)}$ forms a module over $(R, +, \cdot)$.

If $V_1, \cdots, V_r$ and W are unital modules over a unital commutative ring, then so is $\mathrm{Mult}_R(V_1 \times \cdots \times V_r, W)$.

Definition B.14. Let $(V, \oplus, \boxdot)$ be a k-dimensional real vector space equipped with its canonical smooth structure (in the sense that any linear isomorphism $\varphi\colon V \to \mathbb{R}^k$ is a diffeomorphism). Let $v \in V$. For every $w \in V$, the map

$$\gamma^v_w : \mathbb{R} \to M, \quad t \mapsto v \oplus t \boxdot w \tag{B.8}$$

is a (smooth) curve in V. The map

$$\mathcal{J}_v : V \to T_v V, \quad w \mapsto \dot{\gamma}^v_w(0), \tag{B.9}$$

is called the **canonical isomorphism** between V and $T_v V$.

Lemma B.1. *The canonical isomorphism $\mathcal{J}_v : V \to T_v V$ is indeed a linear isomorphism.*

Proof B.1. Denote by $e_1, \ldots, e_k \in V$ a basis of V and by $e^1, \ldots, e^k \in V^*$ its corresponding dual basis. The map

$$\varphi\colon V \to \mathbb{R}^k, \quad v \mapsto \big(e^1(v), \ldots, e^k(v)\big) \tag{B.10}$$

is a global chart for V. As a consequence,

$$\left.\frac{\partial}{\partial e^1}\right|_v, \ldots, \left.\frac{\partial}{\partial e^k}\right|_v \tag{B.11}$$

is a basis of $T_v V$. It holds that

$$\begin{aligned} \mathrm{d}e^i\left(\mathcal{J}_v(w)\right) &= (e^i \circ \gamma^v_w)\dot{}(0) \\ &= e^i(w). \end{aligned} \tag{B.12}$$

This proves that $\mathcal{J}_v : V \to T_v V$ is linear, injective and surjective. This concludes the proof that $\mathcal{J}_v : V \to T_v V$ is a linear isomorphism.

C Group actions

Let $(G, \bullet)$ be a group and X a set.

Definition C.1 (Left action (of a group on a set))**.** A map $\triangleright \colon G \times X \to X$ satisfying

$$\forall x \in X \colon e \triangleright x = x \tag{LGA.1}$$

and

$$\forall g_1, g_2 \in G \colon \forall x \in X \colon g_2 \triangleright (g_1 \triangleright x) = (g_2 \bullet g_1) \triangleright x \tag{LGA.2}$$

is called a **left G-action on X**.

Definition C.2 (Right action (of a group on a set))**.** A map $\triangleleft \colon X \times G \to X$ satisfying

$$\forall x \in X \colon x \triangleleft e = x \tag{RGA.1}$$

and

$$\forall g_1, g_2 \in G \colon \forall x \in X \colon (x \triangleleft g_1) \triangleleft g_2 = x \triangleleft (g_1 \bullet g_2) \tag{RGA.2}$$

is called a **right G-action on X**.

Definition C.3 (Orbit of a point under a group action)**.** If $\triangleright \colon G \times X \to X$ is a left action of G on X we define for any point $x \in X$ its **orbit**

$$\mathrm{Orb}_{\triangleright}(x) := \{y \in X \mid \exists g \in G \colon g \triangleright x = y\} \equiv G \triangleright x. \tag{C.1}$$

The latter notation is motivated by the fact that the orbit $\mathrm{Orb}_{\triangleright}(x)$ coincides with the range of the map $(\triangleright x) \colon G \to X, g \mapsto g \triangleright x$. If $\triangleleft \colon X \times G \to X$ is a right action of G on X we analogously define for any point $x \in X$ its **orbit** according to

$$\mathrm{Orb}_{\triangleleft}(x) := \{y \in X \mid \exists g \in G \colon x \triangleleft g = y\} \equiv x \triangleleft G. \tag{C.2}$$

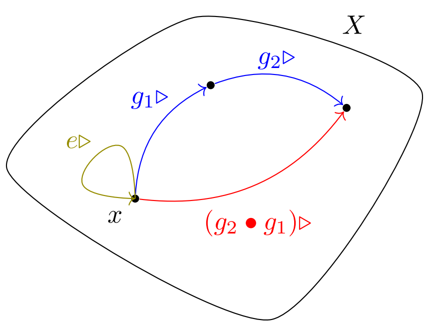

Figure C.1: Defining property of a left action.

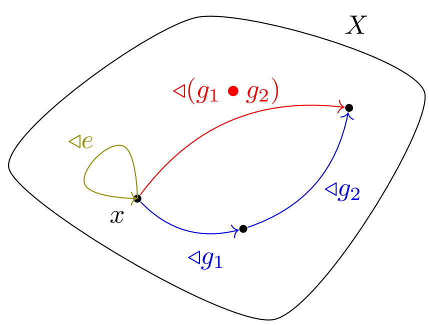

Figure C.2: Defining property of a right action.

Definition C.4 (Stabilizer of a point under a group action)**.** If $\triangleright\colon G \times X \to X$ is a left action of G on X we define for any point $x \in X$ its **stabilizer** given by

$$\mathrm{Stab}_{\triangleright}(x) = \{g \in G \mid g \triangleright x = x\} = (\triangleright x)^{-1}[\{x\}]\,. \tag{C.3}$$

Note that $\mathrm{Stab}_{\triangleright}(x)$ is a subgroup of G due to (LGA.1) and (LGA.2). Analogously, if $\triangleleft\colon X \times G \to X$ is a right action of G on X we define the **stabilizer of** $x \in X$ according to

$$\mathrm{Stab}_{\triangleleft}(x) = \{g \in G \mid x \triangleleft g = x\} = (x\triangleleft)^{-1}[\{x\}]\,. \tag{C.4}$$

Definition C.5 (Effective group action)**.** A left action $\triangleright\colon G \times X \to X$ of G on X is said to be **effective** if the group identity $e \in G$ is the only element that stabilizes every point $x \in X$, i.e., if

$$\bigcap_{x \in X} \mathrm{Stab}_{\triangleright}(x) = \{e\}. \tag{C.5}$$

Analogously we say that a right action $\triangleleft\colon X \times G \to X$ is **effective** if

$$\bigcap_{x \in X} \mathrm{Stab}_{\triangleleft}(x) = \{e\}. \tag{C.6}$$

Definition C.6 (Free group action)**.** A left action $\triangleright\colon G \times X \to X$ of G on X is said to be **free** if for every point $x \in X$ the group identity $e \in G$ is the only element that stabilizes the point x, i.e., if

$$\forall x \in X\colon\ \mathrm{Stab}_{\triangleright}(x) = \{e\}. \tag{C.7}$$

Analogously we say that a right action $\triangleleft\colon X \times G \to X$ is **free** if

$$\forall x \in X\colon\ \mathrm{Stab}_{\triangleleft}(x) = \{e\}. \tag{C.8}$$

Remark C.1. Note that a free group action is always effective. The converse is not true in general.

Definition C.7 (Transitive group action)**.** A left action $\triangleright\colon G \times X \to X$ of G on X is said to be **transitive** if any two points $x, y \in X$ can be joined by the means of a group element $g \in G$ in the sense that $y = g \triangleright x$. This is equivalent to the condition that the orbit of any (or equivalently, some) point $x \in X$ coincides with X, that is,

$$\forall x \in X\colon \quad X = \mathrm{Orb}_{\triangleright}(x) = G \triangleright x. \tag{C.9}$$

Analogously, we say that a right action $\triangleleft\colon X \times G \to X$ is **transitive** if

$$\forall x \in X\colon \quad X = \mathrm{Orb}_{\triangleleft}(x) = x \triangleleft G. \tag{C.10}$$

Proposition C.1 (Free transitive group action induces bijection between group and set)**.** *Show that a transitive transitive right action $\triangleleft\colon X \times G \to X$ yields a bijection $(x\triangleleft)\colon G \to X, g \mapsto x \triangleleft x$ for any point $x \in X$.*

Proof C.1. By the definition of what it means for $\triangleleft$ to be transitive, it is clear that the map $(x\triangleleft)\colon G \to X$ is surjective. Suppose now that for another point $y \in X$ there exist two elements $g, h \in G$ such that $y = x \triangleleft g = x \triangleleft h$. Acting on y with the element g^{-1} yields on the one hand

$$y \triangleleft g^{-1} = (x \triangleleft g) \triangleleft g^{-1} \overset{\text{(RGA.2)}}{=} x \triangleleft \left(g \bullet g^{-1}\right) = x \triangleleft e \overset{\text{(RGA.1)}}{=} x, \tag{C.11}$$

and on the other hand

$$y \triangleleft g^{-1} = (x \triangleleft h) \triangleleft g^{-1} \overset{\text{(RGA.2)}}{=} x \triangleleft \left(h \bullet g^{-1}\right). \tag{C.12}$$

So $x \triangleleft \left(h \bullet g^{-1}\right) = x$, that is, $h \bullet g^{-1}$ stabilizes x. Since $\triangleleft$ is free, it follows that $h \bullet g^{-1} = e$ and thus $h = g$.

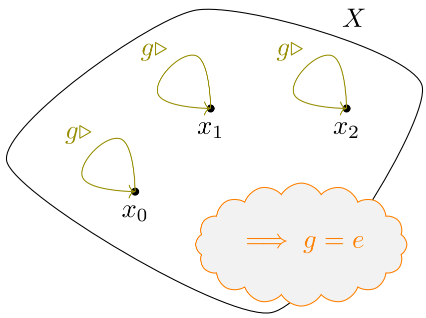

Figure C.3: Defining property of an effective action.

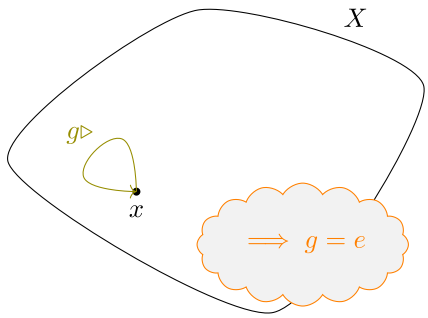

Figure C.4: Defining property of a free action.

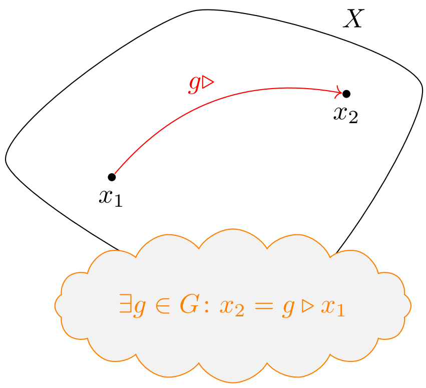

Figure C.5: Defining property of a transitive action.

D Affine spaces

Let $(V, \oplus, \boxdot)$ be a vector space over a field $(\mathbb{K}, +, \cdot)$.

Definition D.1 (Affine space (modelled on a vector space))**.** A non-empty set A together with a free transitive action $\boxplus\colon A \times V \to A$ of the Abelian group $(V, \oplus)$ on A is said to be an **affine space modelled on the vector space** $(V, \oplus, \boxdot)$. Note that $\boxplus\colon A \times V \to A$ is a left and a right action for $(V, \oplus)$ is an Abelian group.

If $(V, \oplus, \boxdot)$ is a n-dimensional vector space over $(\mathbb{K}, +, \cdot)$, we also say that A is an **n-dimensional affine space over the field** $(\mathbb{K}, +, \cdot)$. Note that the dimension is well-defined if finite. We usually denote an **affine space** by the triple $(A, V, \boxplus)$ formed by its underlying set A, the vector space V used to model it and its action $\boxplus$.

Remark D.1 (Subtraction in an affine space). Let $(A, V, \boxplus)$ be an affine space. Since $\boxplus\colon A \times V \to A$ is free and transitive, for any two points $a, b \in A$ there exists a unique $v \in V$ such that $b = a \boxplus v$. Let us call this vector $a \boxminus b \in V$. This defines an operation $\boxminus\colon A \times A \to V$ called the **subtraction in** A. Note that, by definition, for any $a, b \in A$ we have that $b = a \boxplus (b \boxminus a)$. Furthermore, for any $a, b, c \in A$ it holds that $(c \boxminus b) \oplus (b \boxminus a) = c \boxminus a$.

Definition D.2 (Affine map (between affine spaces over a field $(\mathbb{K}, +, \cdot)$))**.** Let $(A, \vec{A}, \boxplus_A)$ and $(B, \vec{B}, \boxplus_B)$ be affine spaces over the same field $(\mathbb{K}, +, \cdot)$.

We say that a map $f\colon A \to B$ is **affine** if there exists a linear map $\vec{f}\colon \vec{A} \to \vec{B}$ such that for any $a \in A$ and $\vec{a} \in \vec{A}$ it holds that

$$f\left(a \underset{A}{\boxplus} \vec{a}\right) = f(a) \underset{B}{\boxplus} \vec{f}(\vec{a}). \tag{D.1}$$

Note that if the above equality holds for some $a \in A$ and all $\vec{a} \in \vec{A}$, then it also holds for all $a \in A$.

Proposition D.1 (Composition of affine maps is affine)**.** *Let $(A, \vec{A}, \boxplus_A)$, $(B, \vec{B}, \boxplus_B)$ and $(C, \vec{C}, \boxplus_C)$ be affine spaces over the same field $(\mathbb{K}, +, \cdot)$ and $f\colon A \to B$ and $g\colon B \to C$ affine maps. Then their composition $g \circ f\colon A \to C$ is affine as well.*

Proof D.1. Let $a \in A$ and $\vec{a} \in \vec{A}$. Since $f\colon A \to B$ is affine, it holds that

$$f\left(a \underset{A}{\boxplus} \vec{a}\right) = f(a) \underset{B}{\boxplus} \vec{f}\,(\vec{a})\,. \tag{D.2}$$

Since $g\colon B \to C$ is affine, it then follows that

$$(g \circ f)\left(a \underset{A}{\boxplus} \vec{a}\right) = g\left(f(a) \underset{B}{\boxplus} \vec{f}\,(\vec{a})\right) = g(f(a)) \underset{C}{\boxplus} \vec{g}\left(\vec{f}\,(\vec{a})\right). \tag{D.3}$$

The composition $\vec{g} \circ \vec{f}\colon \vec{A} \to \vec{C}$ of linear maps is linear. Hence, $g \circ f\colon A \to C$ is indeed an affine map.

Definition D.3 (Affine isomorphism (between affine spaces))**.** Let $(A, \vec{A}, \boxplus_A)$ and $(B, \vec{B}, \boxplus_B)$ be affine spaces over the same field $(\mathbb{K}, +, \cdot)$. A bijective affine map $f\colon A \to B$ whose inverse $f^{-1}\colon B \to A$ is affine as well is said to be an **affine isomorphism**.

Proposition D.2 (Bijective affine map is affine isomorphism)**.** *Let* $(A, \vec{A}, \boxplus_A)$ *and* $(B, \vec{B}, \boxplus_B)$ *be affine spaces over the same field* $(\mathbb{K}, +, \cdot)$*. A bijective affine map* $f\colon A \to B$ *is an affine isomorphism.*

Proof D.2. We will first show that $\vec{f}\colon \vec{A} \to \vec{B}$ is a linear isomorphism. Since $f\colon A \to b$ is affine, equation (D.1) holds for any $a \in A$ and any $\vec{a} \in \vec{A}$. Now fix $a \in A$. Note that $a \boxplus_A \colon \vec{A} \to A$ and $f(a) \boxplus_B \colon \vec{B} \to B$ are bijections due to problem C.1. Therefore the map

$$\left(f(a) \underset{B}{\boxplus} \right)^{-1} \circ f \circ \left(a \underset{A}{\boxplus} \right) : \vec{A} \to \vec{B} \tag{D.4}$$

is a bijection. This map, however, coincides with $\vec{f}\colon \vec{A} \to \vec{B}$. A bijective linear map is a linear isomorphism. As such, $\vec{f}\colon \vec{A} \to \vec{B}$ is a linear isomorphism.

Now it remains to show that $\overrightarrow{f^{-1}}\colon \vec{B} \to \vec{A}$ coincides with the inverse $\vec{f}^{-1}\colon \vec{B} \to \vec{A}$. Let $b \in B$ and $\vec{b} \in \vec{B}$. Then:

$$f^{-1}\left(b \underset{B}{\boxplus} \vec{b} \right) = f^{-1}\left(f\left(f^{-1}(b)\right) \underset{B}{\boxplus} \vec{b} \right) \overset{(D.1)}{=} f^{-1}\left(f\left(f^{-1}(b) \underset{A}{\boxplus} \vec{f}^{-1}(\vec{b}) \right) \right) = f^{-1}(b) \underset{A}{\boxplus} \vec{f}^{-1}(\vec{b}) \tag{D.5}$$

We read off that $\overrightarrow{f^{-1}} = \vec{f}^{-1}$, as claimed.

Remark D.2 (Linear isomorphism induces affine isomorphism)*.* The above problem proves that there exists a linear isomorphism $\vec{f}\colon \vec{A} \to \vec{B}$ for any affine isomorphism $f\colon A \to B$. The converse holds as well. Due to the fact that we required the underlying sets A and B of the affine space to be non-empty, there exist a point $a \in A$ and a point $b \in B$. Then for any linear isomorphism $l\colon \vec{A} \to \vec{B}$ the map

$$\tilde{l}\colon A \to B, \quad a' \mapsto b \underset{B}{\boxplus} \vec{f}\left(a' \underset{A}{\boxminus} a \right) \tag{D.6}$$

is, by definition, an affine isomorphism.

As a consequence, two finite-dimensional affine spaces over the same field belong to the same affine isomorphism class if and only if their dimensions (the dimensions of the vector spaces that they are modelled on) coincide.

Remark D.3 (Representing an affine isomorphism class)*.* If we do not care about the specific representative of an affine isomorphism class, we can resort to a particularly convenient realization of this affine isomorphism class. Suppose we are given an n-dimensional affine space $(A, V, \boxplus)$ over a field $(\mathbb{K}, +, \cdot)$. The n-dimensionality of the underlying vector space V over $\mathbb{K}$ is precisely defined in terms of the cardinality of any Hamel basis, which establishes an isomorphism between V and $\mathbb{K}^n$, the n-th tensor product of $(\mathbb{K}, +, \cdot)$ understood as a vector space over itself.

Observe that the vector space addition $\oplus\colon \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}^n$ in $\mathbb{K}^n$ as the group operation of the Abelian group $(\mathbb{K}^n, \oplus)$ is in particular a free and transitive action of $(\mathbb{K}^n, \oplus)$ on itself. It is thus true that $(\mathbb{K}^n, \mathbb{K}^n, \oplus)$ is an n-dimensional affine space over the field $\mathbb{K}$.

Since there exists a linear isomorphism $\vec{i}\colon V \to \mathbb{K}^n$, there also exists an affine isomorphism $i\colon A \to \mathbb{K}^n$, by D.2. Note that there is, however, no canonical such affine isomorphism.

We conclude that we can always represent the isomorphism class of n-dimensional affine spaces over a field $\mathbb{K}$ by the representative $\mathbb{K}^n$ (regarded as an affine space over $\mathbb{K}$). This is, however, not always conceptually useful. This is due to the lack of a canonical affine isomorphism $i\colon A \to \mathbb{K}^n$. As physicists we would like not to distinguish an arbitrary point of $a \in A$ with the property that $i(a) = 0$.

Definition D.4 (Affine combination)**.** Let $(A, V, \boxplus)$ be an affine space modelled on the vector space $(V, \oplus, \boxdot)$ over a field $(\mathbb{K}, +, \cdot)$. Let $a_1, \ldots, a_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

The **affine combination of the points** $(a_1, \ldots, a_n)$ **with respect to the weights** $(\lambda_1, \ldots, \lambda_n)$ is the point defined by (interpret the left hand side as a newly defined symbol)

$$\sum_{i=1..n}^{\boxplus} \lambda_i a_i := o \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o)\,, \tag{D.7}$$

where $o \in A$ is an arbitrary reference point. It is also called the **barycentre of the points** $(a_1, \ldots, a_n)$ **with respect to the weights** $(\lambda_1, \ldots, \lambda_n)$. Observe that the left hand side is well-defined if and only if $\sum_{i=1}^{n} \lambda_i = 1$.

Proposition D.3 (Affine combination is well-defined)**.** *Let $(A, V, \boxplus)$ be an affine space modelled on the vector space $(V, \oplus, \boxdot)$ over a field $(\mathbb{K}, +, \cdot)$. Let $a_1, \ldots, a_n \in A$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{K}$ such that $\sum_{i=1}^{n} \lambda_i = 1$. Show that the affine combination*

$$\sum_{i=1..n}^{\boxplus} \lambda_i a_i := o \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o) \quad (o \in A) \tag{D.8}$$

is well-defined.

Proof D.3. Let $o, o' \in A$. Then:

$$\begin{aligned}
o' \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o') &\overset{D.1}{=} (o \boxplus (o' \boxminus o)) \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus (o \boxplus (o' \boxminus o))) \\
&= o \boxplus \left((o' \boxminus o) \oplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o') \right) \\
&= o \boxplus \left(\underbrace{\left(\sum_{i=1}^{n} \lambda_i \right)}_{=1} \boxdot (o' \boxminus o) \oplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o') \right) \\
&= o \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot ((a_i \boxminus o') \oplus (o' \boxminus o)) \\
&\overset{D.1}{=} o \boxplus \sum_{i=1..n}^{\oplus} \lambda_i \boxdot (a_i \boxminus o)
\end{aligned} \tag{D.9}$$

Where we used in the second line that $\boxplus \colon A \times V \to A$ is an action of the Abelian group $(V, \oplus)$ and in the fourth line that the distributivity in the vector space $(V, \oplus, \boxdot)$.

Proposition D.4 (Affine map preserves the weights of an affine combination)**.** *Show that an affine map $f \colon A \to B$ between affine spaces $(A, \vec{A}, \boxplus_A)$ and $(B, \vec{B}, \boxplus_B)$ satisfies*

$$f\left(\sum_{i=1..n}^{\boxplus_A} \lambda_i a_i \right) = \sum_{i=1..n}^{\boxplus_B} \lambda_i f(a_i), \tag{D.10}$$

for any affine combination of points $(a_1, \ldots, a_n)$ with respect to weights $(\lambda_1, \ldots, \lambda_n)$.

Proof D.4.

$$
\begin{aligned}
f\left(\sum_{i=1..n}^{\boxplus_A} \lambda_i a_i\right) &\overset{\text{def.}}{=} f\left(o \underset{A}{\boxplus} \sum_{i=1..n}^{\oplus_{\vec{A}}} \lambda_i \underset{\vec{A}}{\boxdot} \left(a_i \underset{A}{\boxminus} o\right)\right) \\
&= f(o) \underset{B}{\boxplus} \vec{f}\left(\sum_{i=1..n}^{\oplus_{\vec{A}}} \lambda_i \underset{\vec{A}}{\boxdot} \left(a_i \underset{A}{\boxminus} o\right)\right) \\
&= f(o) \underset{B}{\boxplus} \sum_{i=1..n}^{\oplus_{\vec{B}}} \lambda_i \underset{\vec{B}}{\boxdot} \vec{f}\left(a_i \underset{A}{\boxminus} o\right) \\
&= f(o) \underset{B}{\boxplus} \sum_{i=1..n}^{\oplus_{\vec{B}}} \lambda_i \underset{\vec{B}}{\boxdot} \left(f(a_i) \underset{B}{\boxminus} f(o)\right) \\
&\overset{\text{def.}}{=} \sum_{i=1..n}^{\boxplus_B} \lambda_i f(a_i)
\end{aligned}
\tag{D.11}
$$

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